EXAM: Numerical Analysis - “Computational Statistics” (CLASS)

[redacted] TIME ALLOWED: 90 minutes

INSTRUCTIONS TO CANDIDATES

1. Always write your name - surname - student ID number - date on **any paper** you are handing in to the instructor or taking a picture of to send via e-mail (`saverio.ranciati2@unibo.it`).
2. this is **NOT** an open book examination: you cannot check books, scripts, lectures materials or any external source of information;
3. candidates may use R and R Studio, together with their built-in “Help”; scripts written during the exam are not to be handed in, so be sure to copy back the relevant code to the paper;
4. if you are taking the exam online, please be sure to write as clearly as possible and that the pictures’ quality is satisfactory.

Question #1: What is the *Probability Integral Transform?* Show how it can be used to simulate data from a distribution with CDF defined as:

$$F(x) = 1 - \exp\{-2^{\frac{1}{2}}\big(\exp\{2x\} - 1\big)\}$$

and provide its `R` implementation.

Question #2: The inverse of the CDF of a Laplace distribution for the random variable $X \sim \text{Laplace}(\mu, b)$, with location $\mu = 3$ and scale $b = 1$, has the following definition

$$F^{-1}(p) = +3 - \text{sign}\Big\{p - \frac{1}{2}\Big\}\ln\Big(1 - 2\Big|p - \frac{1}{2}\Big|\Big).$$

Provide an `R` implementation for:

(A) drawing a sequence of $n = 10^4$ numbers from the Laplace(3,1) using the *Probability Integral Transform*;

(B) drawing another sequence of $n = 10^4$ numbers from the Laplace(0,1), using the relationship

$$E_1 \sim \text{Exp}(\lambda_1), E_2 \sim \text{Exp}(\lambda_2) \rightarrow (\lambda_1 E_1 - \lambda_2 E_2) \sim \text{Laplace}(\mu = 0, b = 1);$$

(C) given that if $X \sim \text{Laplace}(\mu, b) \rightarrow Y = kX + c \sim \text{Laplace}(k\mu + c, kb)$, compare the two results from (A) and (B) by using `R` plots.

Question #3: What is *Importance Sampling*? Describe its structure and use it to provide an estimate for

$$E_f\big[h(x)\big] = \int_{\mathcal{X}} \exp\{-x^2 + 5\}\frac{1}{\pi(1 + x^2)},$$

using as an importance function the distribution $\mathcal{N}(0, 15)$.

(hint: $Y \sim$ Cauchy$(0, 1)$ has density: $\frac{1}{\pi(1+y^2)}$)