VALORE ATTESO e MATRICE DI COVARIANZA

(X, Y), $\begin{pmatrix} X \\ Y \end{pmatrix}$ VALORE ATTESO : $(\mathbb{E}[X], \mathbb{E}[Y])$

$\mathbb{E}[(X, Y)]$

FUNZIONE DI UN VETTORE ALEATORIO

$$h : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$Z = h(X, Y)$$

Teorema

(X, Y) vettore aleatorio discreto con densità discreta congiunta con $p_{(X,Y)}$,

$$\mathbb{E}[Z] = \mathbb{E}[h(X, Y)] = \sum_{i,j} h(x_i, y_j)\, p_{(X,Y)}(x_i, y_j)$$

COROLLARIO

(X,Y) vettore aleatorio discreto (cioè, X e Y sono v.a. discrete).

$$X \perp\!\!\!\perp Y \;\cancel{\Longleftarrow}\!\!\Longrightarrow\; \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y].$$

N.B.

$X \perp\!\!\!\perp Y$: $\mathbb{P}(X \in A,\, Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$, $A, B \subset \mathbb{R}$

$$X \perp\!\!\!\perp Y \iff p_{(X,Y)}(x_i, y_j) = p_X(x_i)\, p_Y(y_j), \ \forall i, \forall j$$

DIM

$$\mathbb{E}[XY] \overset{\text{Thm: } h(x,y) = xy}{=} \sum_{i,j} x_i y_j \, p_{(X,Y)}(x_i, y_j) \overset{X \perp\!\!\!\perp Y}{=} \sum_{i,j} x_i y_j \, p_X(x_i)\, p_Y(y_j)$$

$$= \left(\sum_i x_i p_X(x_i)\right)\left(\sum_j y_j p_Y(y_j)\right) = \mathbb{E}[X]\,\mathbb{E}[Y].$$

OSS.

$$X \perp\!\!\!\perp Y \iff f(X) \perp\!\!\!\perp g(Y)$$

Si può dimostrare che

$$X \perp\!\!\!\perp Y \iff \mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$$

MATRICE DI COVARIANZA

$$\mathbb{E}[(X,Y)] = (\mathbb{E}[X], \mathbb{E}[Y])$$

(X,Y) :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{pmatrix}$$

$(X,Y) = (X_1, X_2)$

$$\begin{pmatrix} cov(X_1, X_1) & cov(X_1, X_2) \\ Cov(X_2, X_1) & cov(X_2, X_2) \end{pmatrix}$$

$$a_{ij} = Cov(X_i, X_j)$$

1) $Cov(X_1, X_1) = Var(X_1)$ | 2) $Cov(X,Y) = Cov(Y,X)$

Definizione

X e Y sono v.a. discrete. La COVARIANZA di X e Y è data da

$$\mathrm{Cov}(X,Y) = \mathbb{E}\Big[\big(X - \mathbb{E}[X]\big)\big(Y - \mathbb{E}[Y]\big)\Big]$$
$$= \sum_{i,j} \big(x_i - \mathbb{E}[X]\big)\big(y_j - \mathbb{E}[Y]\big)\, p_{(X,Y)}(x_i, y_j)$$

Se $\mathrm{cov}(X,Y) = 0$, le v.a. X e Y si dicono SCORRELATE.

Se $\mathrm{Var}(X) > 0$ e $\mathrm{Var}(Y) > 0$, definiamo il COEFFICIENTE DI CORRELAZIONE:

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}.$$

OSS.

1) $x_i - \mathbb{E}[X] =$ scarto dalla media di x_i

$\mathrm{Cov}(X,Y) =$ somma del prodotto degli scarti dalla media

2) $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

3) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$

Teorema 3.2 del Cap. "Stat. descr. e Teoremi limite"

4) $-1 \leq \rho_{X,Y} \leq 1$

Inoltre: $\rho_{X,Y} = \pm 1 \iff Y = aX + b$

Ex. 6

$|\mathrm{Cov}(X,Y)| \leq \sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}$

5) La covarianza ci dice se c'è approssimativamente DIPENDENZA LINEARE.

$\rho_{X,Y} = \pm 1$, allora c'è veramente dipendenza lineare.

$\rho_{X,Y} = 0$, allora non è detto che X e Y siano indipendenti

$$X \perp\!\!\!\perp Y \overset{\not\Longleftarrow}{\Longrightarrow} Cov(X,Y) = 0 \quad (X \text{ e } Y \text{ scorrelate})$$

Teorema

$$X \perp\!\!\!\perp Y \implies \mathrm{Cov}(X,Y) = 0$$

Lemma

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

Dim (Lemma)

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathbb{E}\big[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\big] = \\
&= \mathbb{E}\big[XY - X\,\mathbb{E}[Y] - Y\,\mathbb{E}[X] + \mathbb{E}[X]\,\mathbb{E}[Y]\big] \\
&= \mathbb{E}[XY] - \mathbb{E}[Y]\,\mathbb{E}[X] \cancel{- \mathbb{E}[X]\,\mathbb{E}[Y]} \cancel{+ \mathbb{E}[X]\,\mathbb{E}[Y]}
\end{aligned}$$

N.B.

$$\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$$

Dim (Teorema)

$$X \perp\!\!\!\perp Y \implies \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Quindi

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = $$

$$\underset{X \perp\!\!\!\perp Y}{\uparrow}{=}\ 0$$

$$Y = \underbrace{\log(1+\sin X)}_{Z}$$

Ex. 2.3

X e Y v.a. discrete;

$X \perp\!\!\!\perp Y$, $\quad X \sim B(\frac{1}{2})$, $\quad Y \sim B(\frac{1}{2})$

Siamo $U = X + Y$ e $V = |X - Y|$

a) Determinare congiunta e marginali di U e V.

b) $\mathbb{P}(V < U) = ?$

c) Calcolare $Var(U)$, $Var(V)$, $Cov(U,V)$.

d) U e V sono indipendenti?

a)

$X \backslash Y$	0	1	p_X
0	1/4	1/4	1/2
1	1/4	1/4	1/2
p_Y	$\frac{1}{2}$	$\frac{1}{2}$	1

(X,Y)	(U,V)
$(0,0)$	$(0,0)$
$(0,1)$	$(1,1)$
$(1,0)$	$(1,1)$
$(1,1)$	$(2,0)$

$S_U = \{0,1,2\}$ e $S_V = \{0,1\}$

$U \backslash V$	0	1	p_U
0	1/4	0	1/4
1	0	1/2	1/2
2	1/4	0	1/4
p_V	1/2	1/2	1

$V \sim B(\frac{1}{2})$

$U \sim Bin(2, \frac{1}{2})$

$U = V \mod 2$

b) $\mathbb{P}(V < U) = \sum_{\substack{i,j: \\ v_j < u_i}} p_{(U,V)}(u_i, v_j)$

$= p_{(U,V)}(2,0) + p_{(U,V)}(2,1) = p_U(2) = \frac{1}{4}$

c) $\mathbb{E}[U] = 2 \cdot \frac{1}{2} = 1$

$\mathbb{E}[V] = \frac{1}{2}$

$\mathrm{Var}(U) = 2 \cdot \frac{1}{4} = \frac{1}{2}$

$\mathrm{Var}(V) = \frac{1}{4}$

$\mathbb{E}[UV] = \sum_{i,j} u_i v_j \, p_{(U,V)}(u_i, v_j) = 1 \cdot 1 \cdot p_{(U,V)}(1,1) = \frac{1}{2}$

$\mathrm{Cov}(U,V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V] = \frac{1}{2} - \frac{1}{2} = 0$

d) $U \perp\!\!\!\perp V$?

$U \perp\!\!\!\perp V \iff p_{(U,V)}(u_i, v_j) = p_U(u_i)\, p_V(v_j)$

No, infatti ad esempio

$$p_{(U,V)}(0,0) = \frac{1}{4} \neq \underset{\frac{1}{4}}{\underset{\uparrow}{p_U(0)}}\ \underset{\frac{1}{2}}{\underset{\uparrow}{p_V(0)}}$$