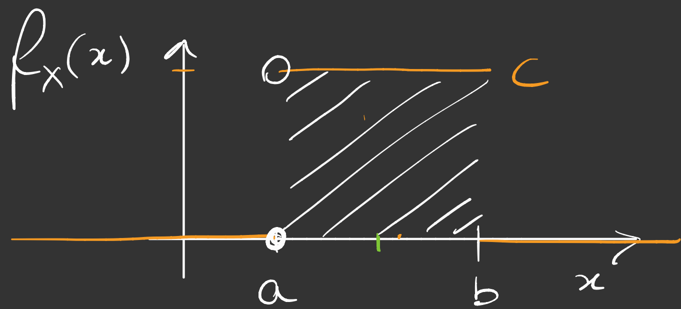
MEDIA e VARIANZA

X v.a.d. (p_X)	X v.a.c. (f_X)
$\mathbb{E}[X] = \sum_i x_i \, p_X(x_i)$ $\sum_i \lvert x_i \rvert \, p_X(x_i) < \infty$	$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \, f_X(x)\,dx$ (x esiste finito) $\int_{-\infty}^{+\infty} \lvert x \rvert f_X(x)\,dx < \infty$
$h: \mathbb{R} \to \mathbb{R}$ $\quad Y = h(X)$ $\mathbb{E}[h(X)] = \sum_i h(x_i)\, p_X(x_i)$	$\mathbb{E}[h(X)] = \int_{-\infty}^{+\infty} h(x) f_X(x)\,dx$

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \sum_i x_i^2 \, p_X(x_i) - \left(\sum_i x_i \, p_X(x_i)\right)^2$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx - \left(\int_{-\infty}^{+\infty} x \, f_X(x)\,dx\right)^2$$

DISTRIBUZIONE UNIFORME (CONTINUA)



$$S_X = [a, b]$$

$$f_X(x) = \begin{cases} 0, & x \notin [a,b] \\ c, & x \in [a,b] \end{cases}$$

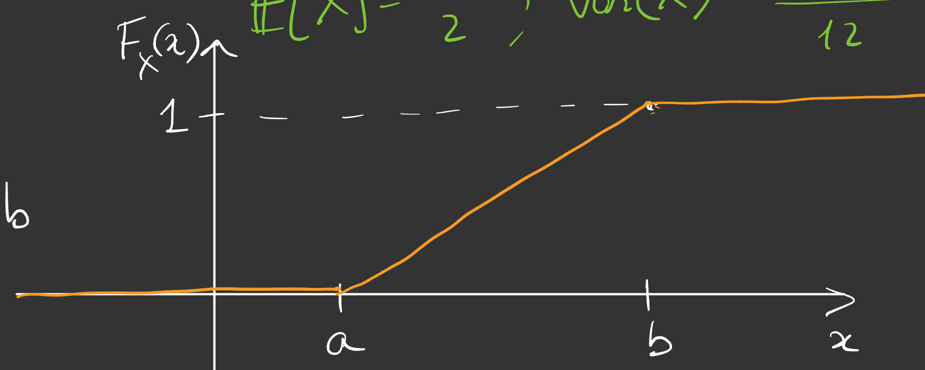
1) $f_X \geq 0 \quad (c \geq 0)$

2) $\displaystyle\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 \iff c = \frac{1}{b-a}$

$$X \sim \mathrm{Unif}(a, b)$$

$$\mathbb{E}[X] = \frac{a+b}{2}\,; \quad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\,dy = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b \end{cases}$$



COME SIMULARE UNA V.A.

Supponiamo di poter simulare un'uniforme

$$X \sim \mathrm{Unif}(0,1)$$

N.B. Se $Z \sim \mathrm{Unif}(a,b)$

$$X = h(Z) = \frac{Z - a}{b-a} \sim \mathrm{Unif}(0,1)$$

$$F_X(x) \underset{\substack{\uparrow \\ 0 \le x \le 1}}{=} \mathbb{P}(X \le x) = \mathbb{P}\left(\frac{Z-a}{b-a} \le x\right) = \mathbb{P}(Z \le x(b-a)+a)$$

$$= F_Z(x(b-a)+a) = \frac{x(b-a)+a-a}{b-a} = x$$

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$

V.A. CONTINUA CON F_Y invertibile (NORMALE o GAUSSIANA)

$\hookrightarrow S_Y = \mathbb{R}$

$$Y = F_Y^{-1}(X), \quad \text{dove} \quad X \sim \text{Unif}(0,1)$$

$x_1, x_2, \dots, x_n$ con legge $\text{Unif}(0,1)$

$y_1 = F_Y^{-1}(x_1), \dots, y_n = F_Y^{-1}(x_n)$

Perché Y ha CDF F_Y?

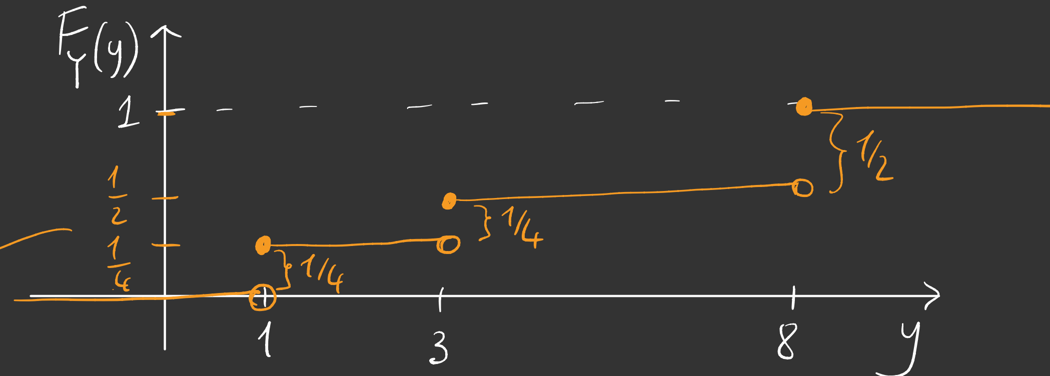
$$\mathbb{P}(Y \leq y) = \mathbb{P}(F_Y^{-1}(X) \leq y) = \mathbb{P}(X \leq F_Y(y)) =$$

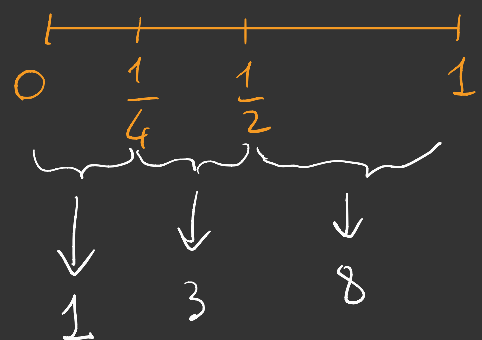
$$= F_X(F_Y(y)) = F_Y(y).$$

V.A. DISCRETA

Y v.a. discreta, trovare $h : \mathbb{R} \to \mathbb{R}$ tale che $Y = h(X)$, con $X \sim \text{Unif}(0,1)$

Y	1	3	8
p_Y	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$





$$h(x) = \begin{cases} 1, & 0 \le x < \frac{1}{4} \\ 3, & \frac{1}{4} \le x < \frac{1}{2} \\ 8, & \frac{1}{2} \le x < 1 \end{cases}$$

$x_1, x_2, \dots, x_n$

$y_1 = h(x_1), y_2 = h(x_2), \dots, y_n = h(x_n)$

25% in $[0, \frac{1}{4}]$

25% in $[\frac{1}{4}, \frac{1}{2}]$

50% in $[\frac{1}{2}, 1]$

25% di numeri uguali a 1 ———— 3

50% ———— 8

COME GENERARE Unif$([0,1])$

generatori lineari congruenziali (LCG)

$$x_n = (a x_{n-1} + c) \bmod m$$

NUMERI PSEUDOCASUALI

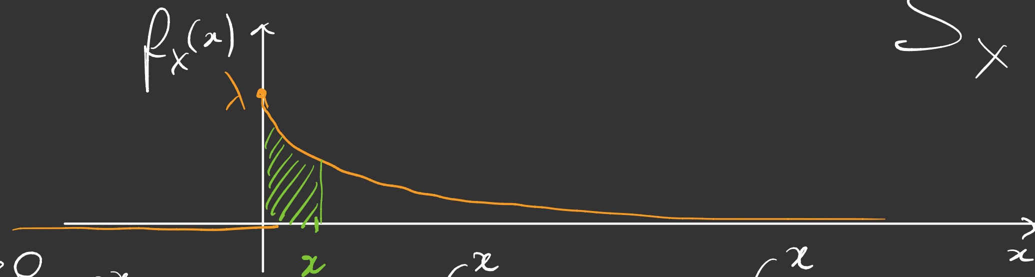
$$m = 2^{31} - 1$$

DISTRIBUZIONE ESPONENZIALE

X è v.a. continua con densità

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases} \qquad \lambda > 0$$

$$S_X = [0, +\infty)$$

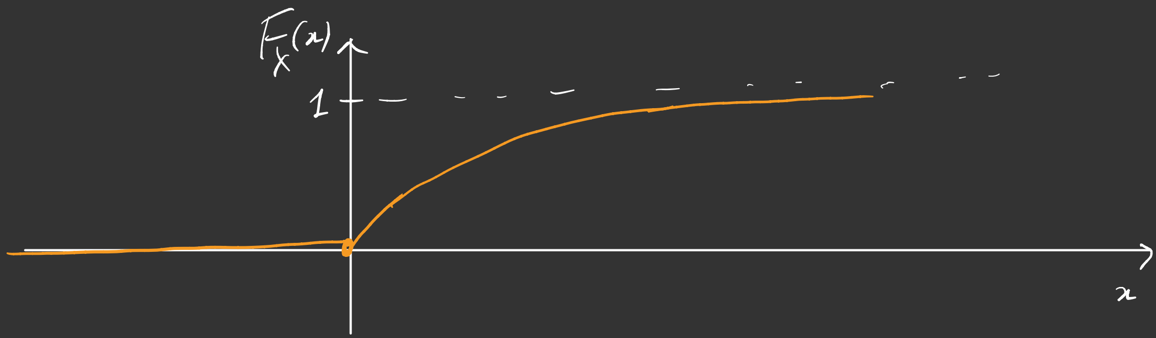


$$F_X(x) \overset{x>0}{=} \int_{-\infty}^{x} f_X(y)\,dy = \int_0^x f_X(y)\,dy = \int_0^x \lambda e^{-\lambda y}\,dy = \left[-e^{-\lambda y}\right]_0^x =$$

$$= -e^{-\lambda x} - (-e^0) = 1 - e^{-\lambda x}, \qquad x \geq 0$$

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

$X \sim \mathrm{Exp}(\lambda)$



$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_0^{+\infty} x \lambda e^{-\lambda x}\,dx =$$

$$= \underbrace{\left[x\left(-e^{-\lambda x}\right)\right]_0^{+\infty}}_{0-0} - \int_0^{+\infty} 1\left(-e^{-\lambda x}\right)dx =$$

$$= 0 + \frac{1}{\lambda}\underbrace{\int_0^{+\infty} \lambda e^{-\lambda x}\,dx}_{=1} = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_0^{+\infty} x^2 \lambda e^{-\lambda x} dx =$$

$$= \underbrace{\left[x^2(-e^{-\lambda x})\right]_0^{+\infty}}_{0\,-\,0} - \int_0^{+\infty} 2x(-e^{-\lambda x})\,dx =$$

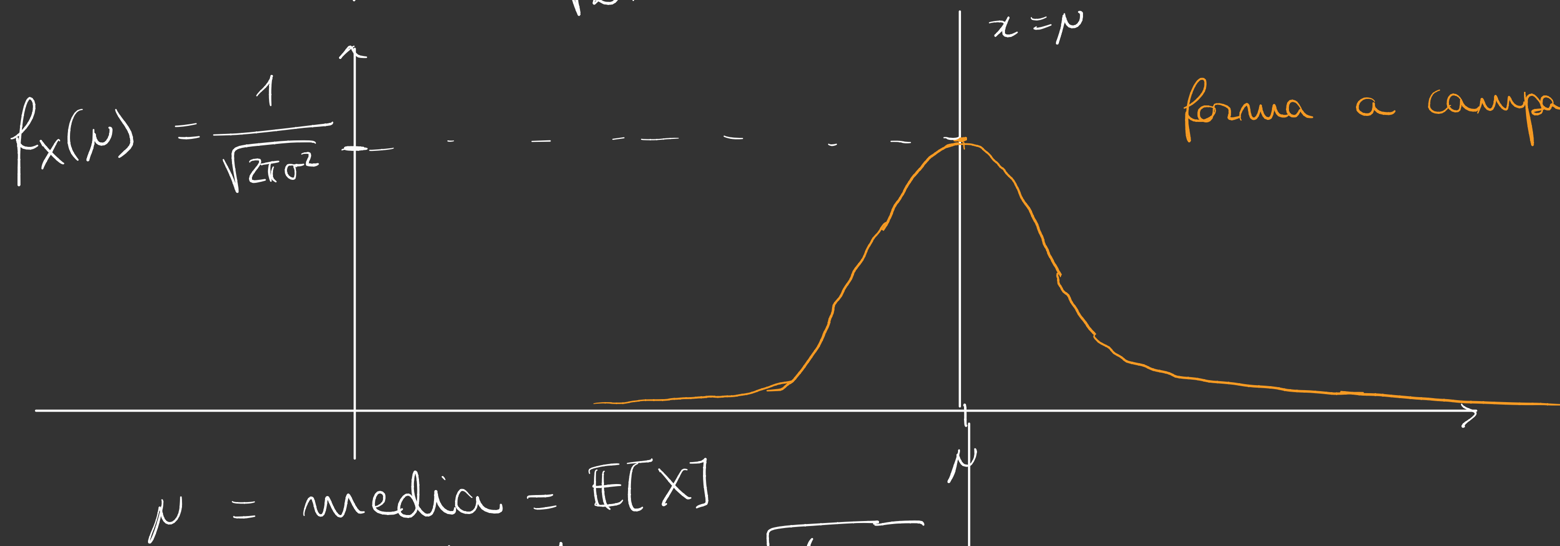
$$= 0 + \frac{2}{\lambda} \underbrace{\int_0^{+\infty} x \lambda e^{-\lambda x} dx}_{\mathbb{E}[X] = \frac{1}{\lambda}} = \frac{2}{\lambda^2}$$

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

DISTRIBUZIONE NORMALE o GAUSSIANA

X ha distribuzione normale di parametri $\mu \in \mathbb{R}$ e $\sigma > 0$ se X v.a. continua con densità

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}, \quad \forall x \in \mathbb{R}.$$



μ = media = $\mathbb{E}[X]$

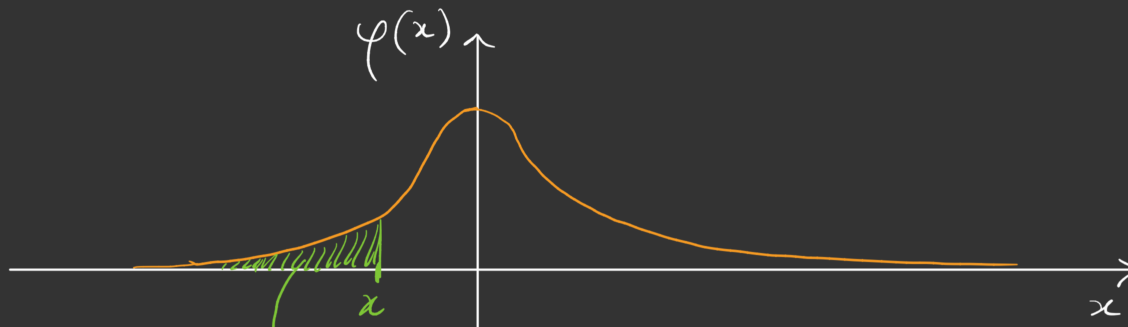
σ = dev. standard = $\sqrt{Var(X)}$

$$X \sim N(\mu, \sigma^2)$$

NORMALE <u>STANDARD</u> : $N(0,1) \sim Z$

$$f_Z(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$F_Z(x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy$$



$$F_Z(x) = \Phi(x)$$

STANDARDIZZAZIONE

$$X \sim N(\mu, \sigma^2) \implies Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$F_Z(x) = \mathbb{P}(Z \leq x) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq x\right) = \mathbb{P}(X \leq \sigma x + \mu) =$$

$$= F_X(\sigma x + \mu)$$

$$F_Z(x) = F_X(\sigma x + \mu)$$

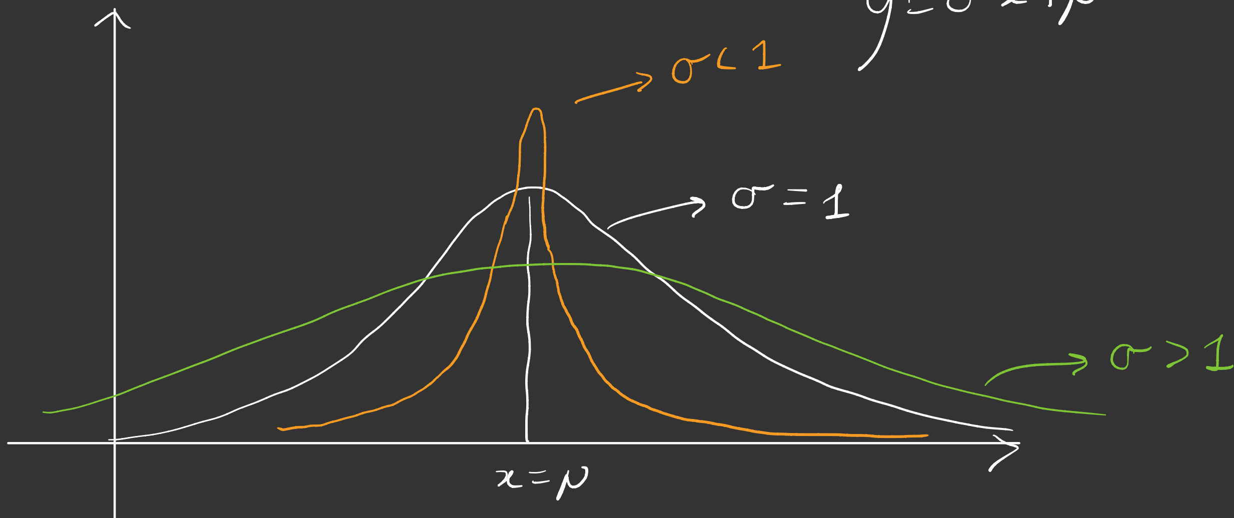
$$\downarrow$$

$$f_Z(x) = f_X(\sigma x + \mu) \cdot \sigma = \frac{1}{\sqrt{2\pi \cancel{\sigma^2}}} e^{-\frac{1}{2}\frac{(\sigma x + \mu - \mu)^2}{\sigma^2}} \cdot \cancel{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \varphi(x)$$

$$\underbrace{F_Z(x)}_{\Phi(x)} = F_X(\sigma x + \mu) \iff F_X(y) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$y = \sigma x + \mu$$



OSS.

$$I = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$$

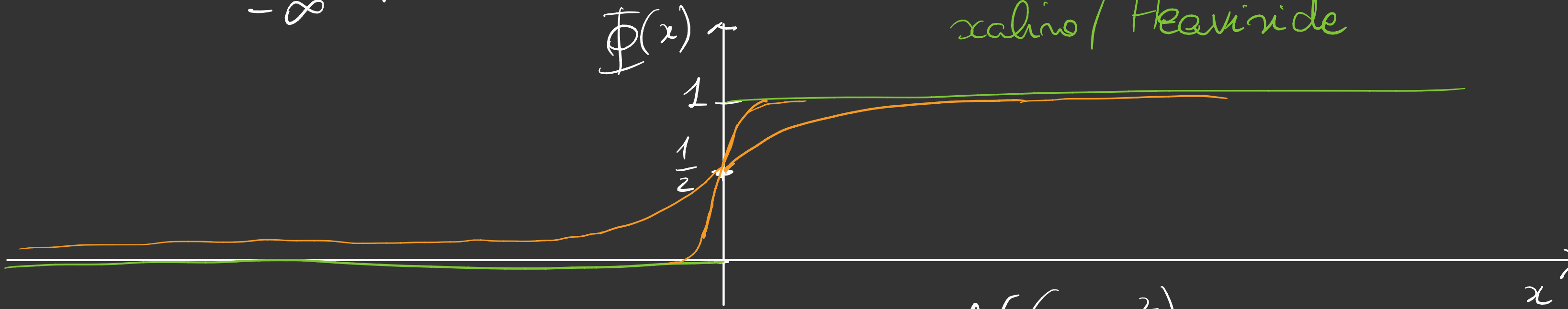
(INTEGRALE DI GAUSS)

$$I^2 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dx$$

PROPOSIZIONE (PROPRIETÀ DI Φ)

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy, \qquad \forall x \in \mathbb{R}$$



$N(0, \sigma^2)$

$\sigma < 1$

1) $\Phi(0) = \frac{1}{2}$

2) $\Phi(-x) = -\Phi(x) + 1$

1) $\Phi(0) = \frac{1}{2}$

In 2) prendo $x = 0 \implies$ 1)

2) $\Phi(-x) = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \underset{\substack{\uparrow \\ z=-y}}{=} \int_{x}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz =$

$= \mathbb{P}(Z \geq x) = 1 - \mathbb{P}(Z \leq x) =$

$= 1 - \Phi(x)$