CARATTERIZZAZIONE DELLE V. A. DISCRETE

TEOREMA

$X : \Omega \to \mathbb{R}$ variabile aleatoria. Queste affermazioni sono tra loro equivalenti:

1) X è v.a DISCRETA, ossia $\exists\, S_X \subset \mathbb{R}$, finito o infinito numerabile, tale che:

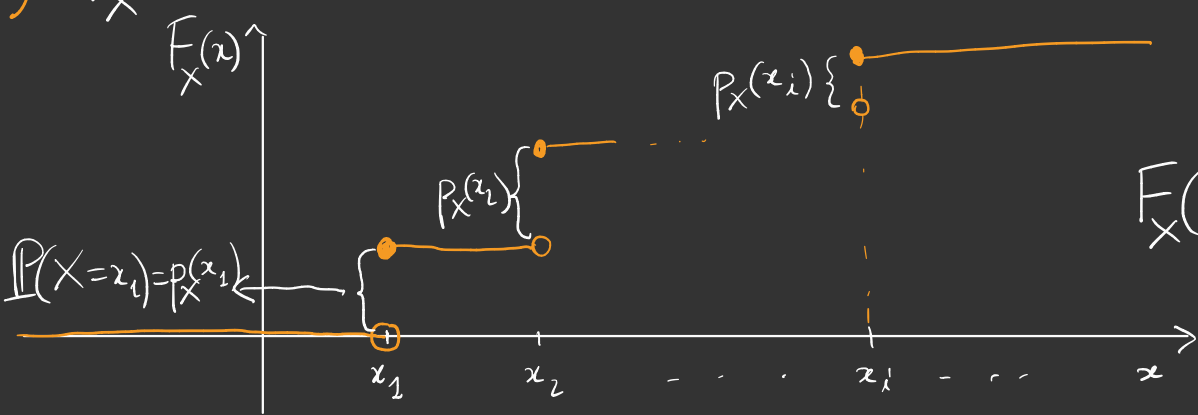
a) $\mathbb{P}(X = x_i) > 0, \quad \forall x_i \in S_X$

b) $\mathbb{P}(X \in S_X) = \sum_i \mathbb{P}(X = x_i) = 1$

a) $p_X(x_i) > 0$

b) $\sum_i p_X(x_i) = 1$

2) F_X è una funzione COSTANTE A TRATTI



$$F_X(x_i) = F_X(x_i^-) + p_X(x_i)$$

3) $\mathbb{P}_X$ è concentrata nei punti $x_1, \dots, x_i, \dots$ di S_X :

$$\mathbb{P}_X(B) = \sum_i p_X(x_i)\delta_{x_i}(B) \qquad \forall B \subset \mathbb{R}.$$

N.B.

$$\mathbb{P}(X \in B) = \sum_i p_X(x_i)\delta_{x_i}(B) =$$

$$= \sum_{i;\, x_i \in B} p_X(x_i)$$

MEDIA e VARIANZA

(INDICI DI SINTESI DI UNA DISTRIBUZIONE)

Definizione

X v.a. discreta, la media o valore atteso è data da

$$\mathbb{E}[X] = \sum_i x_i \, p_X(x_i) \qquad x_i \in S_X$$

(μ_X)

OSS. 1

$$\mathbb{E}[\cdot] : X \longmapsto \mathbb{E}[X] \in \mathbb{R}$$

OSS. 2 Se S_X è infinito numerabile il valore atteso è definito solo se la serie $\sum_i x_i \, p_X(x_i)$ è ass. conv., cioè

$$\sum_i |x_i| \, p_X(x_i) < \infty$$

V.A. COSTANTI

$a \in \mathbb{R}$ fissata e $X(\omega) = a, \quad \forall \omega \in \Omega$

X	a
p_X	1

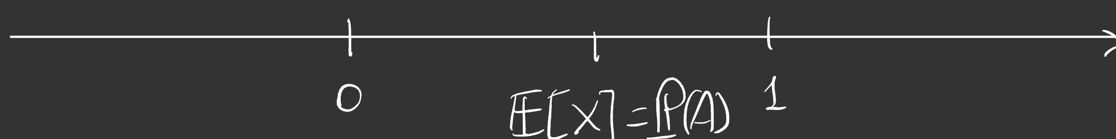
$$\mathbb{E}[X] = \mathbb{E}[a] = a \cdot 1$$

V.A. INDICATRICI

$A \subset \Omega$ e $X = \mathbb{1}_A$

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{1}_A] = \cancel{0 \cdot (1 - \mathbb{P}(A))} + 1 \cdot \mathbb{P}(A)$$

$$= \mathbb{P}(A)$$



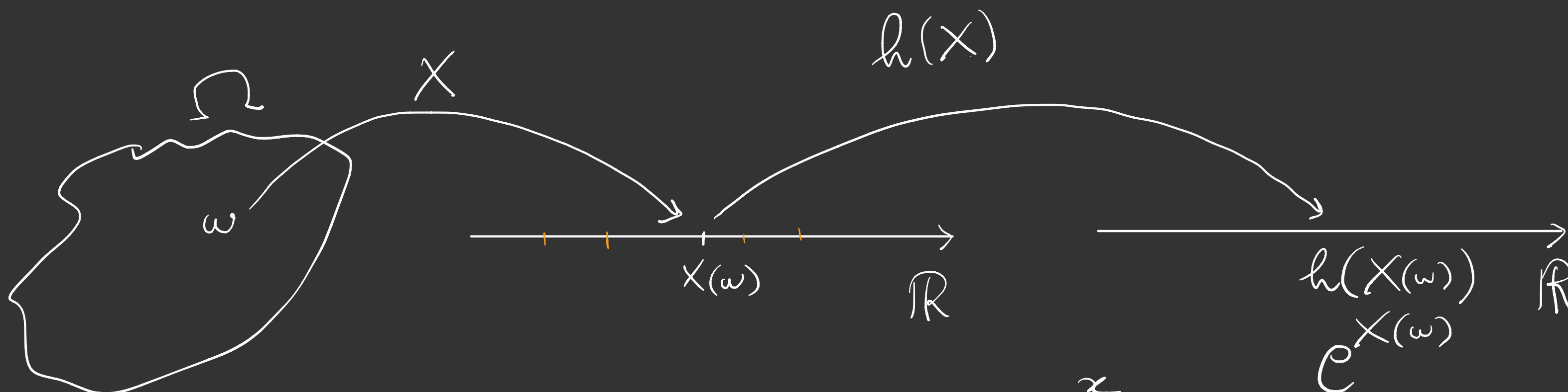
TEOREMA

X v.a. discreta e $h : \mathbb{R} \to \mathbb{R}$. Allora consideriamo la v.a.

$$Y = h(X)$$

Vale che

$$\mathbb{E}[Y] = \mathbb{E}[h(X)] = \sum_i h(x_i)\, p_X(x_i).$$



$$h(x) = e^x$$

DIM.

$$\begin{array}{ccc} x_1 & & h(x_1) \\ x_2 & \xrightarrow{\quad h \quad} & h(x_2) \\ \vdots & & \vdots \\ x_i & & h(x_i) \\ \vdots & & \\ S_X & & S_Y \end{array}$$

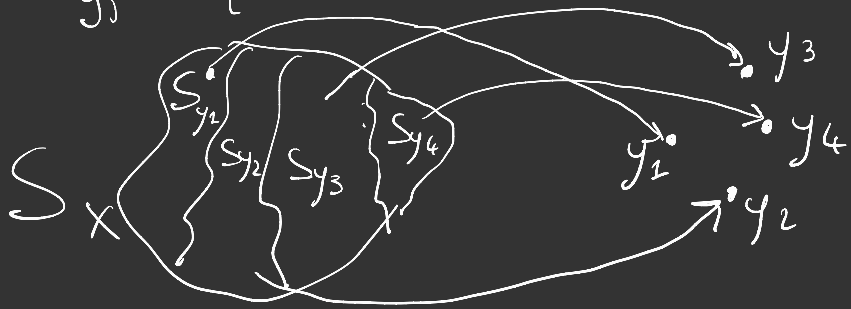
S_Y è finito o infinito numerabile

$|S_Y| \leq |S_X|$

$S_X = \{x_1, \dots, x_n\}$ FINITO

$S_Y = \{y_1, \dots, y_m\}$ $\quad m \leq n$

$S_{y_j} = \{x \in S_X : h(x) = y_j\}$



$$\mathbb{E}[Y] := \sum_{j=1}^{m} y_j \, p_Y(y_j) = \sum_{j=1}^{m} y_j \, \mathbb{P}(Y = y_j) = (*)$$

$$(Y = y_j) = \bigcup_{\substack{i: \\ x_i \in S_{y_j}}} (X = x_i) = \bigcup_{\substack{i: \\ h(x_i) = y_j}} (X = x_i)$$

$$(*) = \sum_{j=1}^{m} y_j \left(\sum_{\substack{i: \\ x_i \in S_{y_j}}} p_X(x_i) \right) =$$

$$= \sum_{j=1}^{m} \left(\sum_{\substack{i: \\ x_i \in S_{y_j}}} h(x_i) \, p_X(x_i) \right) = \sum_{i=1}^{n} h(x_i) \, p_X(x_i)$$

TEOREMA (Linearità del valore atteso)

X v.a. discreta, $a, b \in \mathbb{R}$

$$\mathbb{E}[aX+b] = a\,\mathbb{E}[X] + b$$

DIM $h(x) = ax + b$,

$$\mathbb{E}[aX+b] = \mathbb{E}[h(X)] \overset{\text{Teorema}}{=} \sum_i h(x_i)\, p_X(x_i) =$$

$$= \sum_i (a x_i + b)\, p_X(x_i) =$$

$$= a \sum_i x_i\, p_X(x_i) + b \underbrace{\sum_i p_X(x_i)}_{=1}$$

$$= a\,\mathbb{E}[X] + b.$$

Definizione

X v.a. discreta. La VARIANZA di X è data da

$$\mathrm{Var}(X) := \mathbb{E}\Big[\underbrace{(X - \mathbb{E}[X])^2}_{h(X)}\Big] \underset{\text{Teorema}}{=} \sum_i (x_i - \mathbb{E}[X])^2 p_X(x_i)$$

σ^2

$$\sigma = \sqrt{\mathrm{Var}(X)} = $$ deviazione standard

scostamento quadratico medio

scarto

Scostamento dalla media: $x_i - \mathbb{E}[X]$

$$\sum_i |x_i - \mathbb{E}[X]|\, p_X(x_i)$$

TEOREMA

X v.a. discreta:

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 =$$

$$\underset{\substack{\uparrow \\ \text{Teorema}}}{=} \sum_i x_i^2 p_X(x_i) - \left(\sum_i x_i p_X(x_i)\right)^2$$

DIM

$$\mathrm{Var}(X) := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\right]$$

$$\underset{\substack{\uparrow \\ \text{LINEARITÀ}}}{=} \mathbb{E}[X^2] - 2\underbrace{\mathbb{E}[X]\,\mathbb{E}[X]}_{\mathbb{E}[X]^2} + \mathbb{E}[X]^2 =$$

$$= \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

v.a., $(X - \mathbb{E}[X])^2$, cioè

$$\omega \longmapsto (X(\omega) - \mathbb{E}[X])^2 = Y(\omega)$$

Teorema

1) $Var(X) \geq 0$ $\quad \left(\sum_i (x_i - \mathbb{E}[X])^2 p_X(x_i) \geq 0 \right)$

2) $Var(b) = 0$, se $b \in \mathbb{R}$ fissata; viceversa:

$$Var(X) = 0 \iff X \text{ v.a. costante } \left(X = \mathbb{E}[X] \right)$$

3) $Var(aX+b) = a^2 \, Var(X)$

$$Var(aX+b) = \sum_i \Big(a x_i + \cancel{b} - \underbrace{\mathbb{E}[aX+b]}_{a\mathbb{E}[X] + \cancel{b}}\Big)^2 p_X(x_i) =$$

$$= a^2 \sum_i (x_i - \mathbb{E}[X])^2 p_X(x_i) = a^2 \, Var(X).$$

2) $\mathrm{Var}(X) = 0 \iff X$ costante.

$\Leftarrow$

$X = b \implies \mathrm{Var}(b) = \mathbb{E}[(b - \mathbb{E}[b])^2] = 0$

$\Rightarrow$

$\mathrm{Var}(X) = 0 \implies \sum_i (x_i - \mathbb{E}[X]) p_X(x_i) = 0$

$\implies \forall i : (x_i - \mathbb{E}[X]) p_X(x_i) = 0$

$\implies x_i - \mathbb{E}[X] = 0$

$\implies X = \mathbb{E}[X].$

DISTRIBUZIONE UNIFORME DISCRETA

$$S_X = \{x_1, \dots, x_n\}$$

X	x_1	$\dots$	x_n
p_X	$\frac{1}{n}$		$\frac{1}{n}$

Lancio del dado

X = "risultato del lancio"

X	1	2	3	4	5	6
p_X	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$X \sim Unif(\{1, 2, 3, 4, 5, 6\})$$

$$\mathbb{P}_X = Unif(\{x_1, \dots, x_n\})$$

$$\mathbb{E}[X] = \frac{x_1 + \dots + x_n}{n} \quad e \quad Var(X) = \frac{(x_1 - \mathbb{E}[X])^2 + \dots + (x_n - \mathbb{E}[X])^2}{n}$$

DISTRIBUZIONE DI BERNOULLI

$X = \mathbb{1}_A$, $\quad A \subset \Omega \quad A =$ "esce testa"

X	0	1
P_X	$1-p$	p

$p := \mathbb{P}(A) \qquad X \sim B(p)$

$$\mathbb{P}_X = B(p)$$

$\mathbb{E}[X] = p$ e $\mathrm{Var}(X) = p(1-p) =$

$= \mathbb{E}[X^2] - \mathbb{E}[X]^2 =$

$= 0^2 \cdot (1-p) + 1^2 \cdot p - p^2 =$

$= p(1-p)$

DISTRIBUZIONE BINOMIALE

n prove di Bernoulli indipendenti e con la stessa probabilità di successo p.

$$A_i = \text{"successo all'i-esima prova"}$$

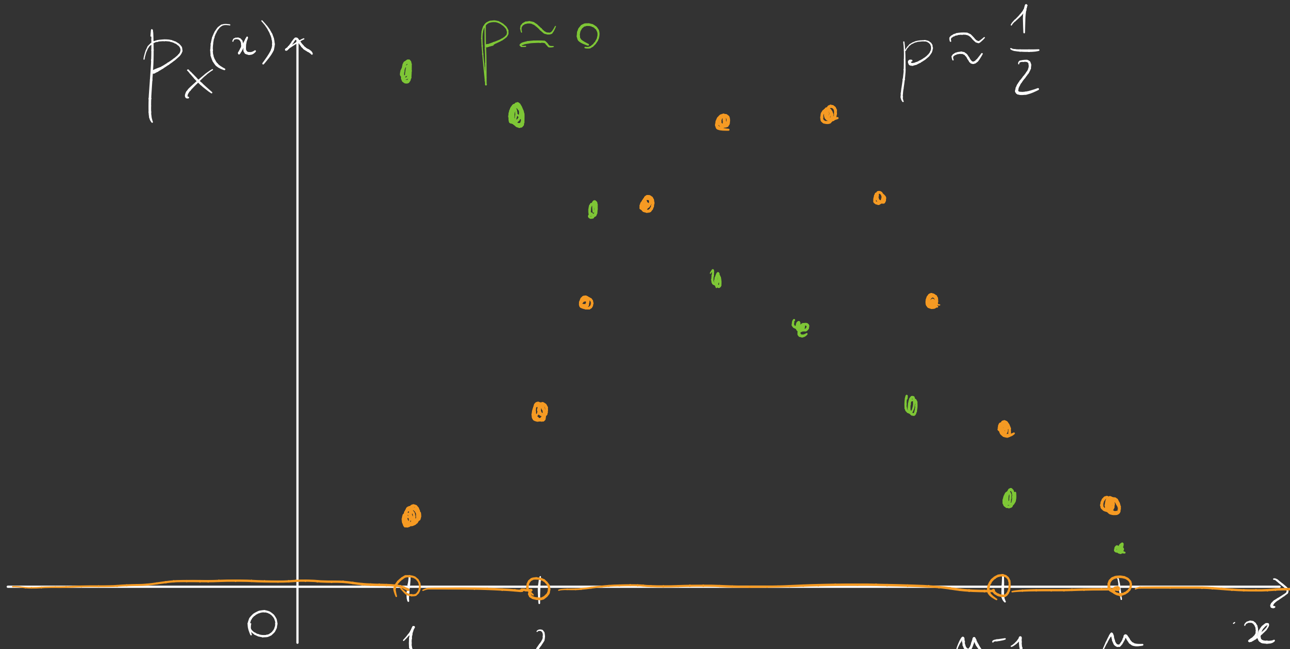
$$X_i = \mathbb{1}_{A_i} = \begin{cases} 1, & \text{successo all'i-esima prova} \\ 0, & \text{insuccesso} \end{cases}$$

$$X = X_1 + \dots + X_n = \text{"n° di successi in n prove"}$$

$$S_X = \{0, 1, 2, \dots, n\}$$

$$p_X(k) = \mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, \dots, n$$

X	0	1	$\dots$	$n-1$	n
p_X	$(1-p)^n$	$np(1-p)^{n-1}$		$np^{n-1}(1-p)$	p^n



$$\mathbb{P}_X = B(n,p) \qquad X \sim B(n,p)$$

$$\mathbb{E}[X] = np = \mathbb{E}[X_1 + \dots + X_n]$$

$$\mathbb{E}[X] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot k =$$

$$= \sum_{k=\cancel{0}\,1}^{n} \frac{n!}{\underset{(k-1)!}{\cancel{k!}}\,(n-k)!}\, p^k (1-p)^{n-k} \cancel{k} =$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} =$$

$h = k-1$

$$= np \sum_{h=0}^{n-1} \binom{n-1}{h} p^h (1-p)^{n-1-h} \to (p+1-p)^{n-1} = 1$$

$$\mathrm{Var}(X) = np(1-p) = \mathrm{Var}(X_1 + \dots + X_n) =$$

$$\underset{\text{indip.}}{=} \mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_n)$$