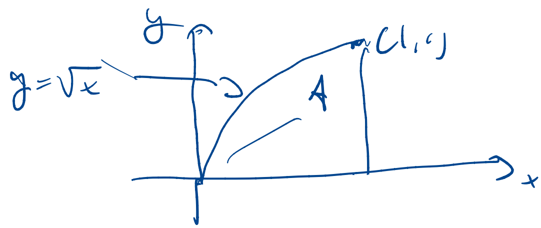
ES 1.1 $\quad A = \{ (x,y) \in \mathbb{R}^2 \mid x \in [0,1],\ 0 \le y \le \sqrt{x} \}$

$0 \le y \le \sqrt{x}$

Applichiamo la formula di riduzione



$$\int_A \frac{y}{x+y^2+1}\,dx\,dy =$$

$$= \int_0^1 \left(\int_0^{\sqrt{x}} \frac{y}{x+y^2+1}\,dy \right) dx = \int_0^1 \frac{1}{2}\Big[\ln(x+y^2+1)\Big]_{y=0}^{y=\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^1 \big(\ln(1+2x) - \ln(1+x) \big)\, dx = (a) + (b)$$

$$\int_0^1 \ln(1+2x)\,dx = \left[\left(\frac{1+2x}{2}\right)\ln(1+2x)\right] - \int_0^1 \frac{1+2x}{2}\cdot\frac{2}{1+2x}\,dx$$

$$= \left[\left(\frac{1+2x}{2}\right)\ln(1+2x) - x\right]_0^1 = \frac{3}{2}\ln 3 - 1.$$

$$\int_0^1 \ln(1+x)\,dx = \Big[(1+x)\ln(1+x)\Big]_0^1 - \int_0^1 (1+x)\cdot\frac{1}{1+x}\,dx$$

$$= 2\ln 2 - 1$$

Quindi $(a)+(b) = \frac{1}{2}\left(\frac{3}{2}\ln 3 - 1 - (2\ln 2 - 1)\right)$

$$= \frac{3}{4}\ln 3 - \ln 2$$

In alternativa si possono calcolare (a) e (b) ponendo $t = 1+2x$ e $t = 1+x$ ...

Made with Goodnotes

ES 2 $f(x,y) = x^3 - 4\left(x - \frac{1}{2}\right)(y - y^2)$

$$\begin{cases} \partial_x f = 3x^2 - 4(y - y^2) = 0 \\ \partial_y f = -4\left(x - \frac{1}{2}\right)(1 - 2y) = 0 \end{cases}$$

Dalla seconda equazione ottengo $x = \frac{1}{2}$ oppure $y = \frac{1}{2}$.

$$\begin{cases} x = \frac{1}{2} \\ \partial_x f\left(\frac{1}{2}, y\right) = \frac{3}{4} - 4(y - y^2) = \frac{1}{4}(16y^2 - 16y + 3) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 1/2 \\ y = \frac{8 \pm \sqrt{64 - 48}}{16} = \frac{8 \pm 4}{16} = \begin{cases} 3/4 \\ 1/4 \end{cases} \end{cases}$$

Si ottengono i punti critici

$P_1 = \left(\frac{1}{2}, \frac{3}{4}\right)$ e $P_2 = \left(\frac{1}{2}, \frac{1}{4}\right)$.

Se invece $y = \frac{1}{2}$, risulta

$$\begin{cases} y = \frac{1}{2} \\ 3x^2 - 4\left(\frac{1}{2} - \frac{1}{4}\right) = 0 \end{cases} \Leftrightarrow \begin{cases} y = 1/2 \\ 3x^2 = 1 \end{cases} \Rightarrow \begin{matrix} P_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{2}\right) \\ P_4 = \left(-\frac{1}{\sqrt{3}}, \frac{1}{2}\right) \end{matrix}$$

Calcolo la matrice hessiana

$$Hf = \begin{bmatrix} 6x & -4(1 - 2y) \\ -4(1-2y) & 8\left(x - \frac{1}{2}\right) \end{bmatrix}$$

$$Hf\left(\frac{1}{2}, \frac{3}{4}\right) = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$ Ha $\det < 0$. $\Rightarrow \left(\frac{1}{2}, \frac{3}{4}\right)$ è di sella

$$Hf\left(\frac{1}{2}, \frac{1}{4}\right) = \begin{bmatrix} 3 & -2 \\ -2 & 0 \end{bmatrix}$$ Ha ancora $\det < 0$
$\Rightarrow \left(\frac{1}{2}, \frac{1}{4}\right)$ è di sella

Made with Goodnotes

$$Hf\left(\frac{1}{\sqrt{3}}, \frac{1}{2}\right) = \begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & 8\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right) \end{bmatrix}$$

Essendo $\frac{1}{\sqrt{3}} - \frac{1}{2} > 0$, il punto è di minimo

$$Hf\left(-\frac{1}{\sqrt{3}}, \frac{1}{2}\right) = \begin{bmatrix} -3\sqrt{3} & 0 \\ 0 & 8\left(-\frac{1}{\sqrt{3}} - \frac{1}{2}\right) \end{bmatrix}$$

Il punto è di massimo locale (matrice diagonale a elementi negativi).

Made with Goodnotes

ES 3 È noto che

$$\nabla f(x,y) = (y, x + \sin y) \; ; \; r(t) = (\sin(2t), 2t)$$

$$r'(t) = (2\cos(2t), 2).$$

Allora la formula per la derivata lungo una curva dà:

$$\frac{d}{dt} f(r(t)) = \langle \nabla f(r(t)), r'(t) \rangle$$

$$= \langle (2t, \sin(2t) + \sin(2t)), (2\cos(2t), 2) \rangle$$

$$= 4t\cos(2t) + 4\sin(2t)$$

Made with Goodnotes