10. Dicembre. 2021

ESERCIZIO VI

$$\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x \cdot \sin x} \right) \qquad [+\infty - \infty]$$

$$\frac{1}{x^2} - \frac{1}{x \cdot \sin x} = \frac{\sin x - x}{x^2 \cdot \sin x}$$

$$x^2 \cdot \sin x = x^2 \left[x + o(x) \right] =$$

$$= x^3 + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\frac{1}{x^2} - \frac{1}{x \cdot \sin x} = \frac{-\frac{x^3}{6} + o(x^3)}{x^3 + o(x^3)} = \frac{-\frac{1}{6} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} \xrightarrow{x \to 0} -\frac{1}{6}$$

ESERCIZIO VII :

$$\lim_{x\to 0} (\cos x)^{-\frac{1}{x^2}} \qquad [1^{-\infty}]$$

$$(\cos x)^{-\frac{1}{x^2}} = e^{\ln(\cos x)^{-\frac{1}{x^2}}} =$$

$$= e^{-\frac{1}{x^2}\ln\cos x} = e^{-\frac{\ln\cos x}{x^2}}$$

$$\lim_{x\to 0} -\frac{\ln\cos x}{x^2} = ?$$

si deve sviluppare il $\ln(\cos x)$ al II ordine :

$$\ln\cos x = \ln\Big(1 + \underbrace{\boxed{\left(-\frac{x^2}{2} + o(x^2)\right)}}_{=t}\Big)$$

$$t \approx x^2 \Rightarrow o(t) = o(x^2)$$

$$\ln(1+t) = t + o(t) = -\frac{x^2}{2} + o(x^2)$$

$$\ln \cos x = -\frac{x^2}{2} + o(x^2)$$

$$\lim_{x\to 0} -\frac{\ln \cos x}{x^2} = \lim_{x\to 0} -\frac{-\frac{x^2}{2} + o(x^2)}{x^2} =$$

$$= \lim_{x\to 0} +\frac{1}{2} - \frac{o(x^2)}{x^2} = -\frac{1}{2}$$

$\Rightarrow$

$$\lim_{x\to 0} (\cos x)^{-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0} e^{-\frac{\ln \cos x}{x^2}} = e^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{e}}$$

Esercizio $\underline{VIII}$:

$$\lim_{x\to 0} \frac{\ln(\cos x) + \frac{1}{2}e^{x^2} - \frac{1}{2}}{x^4} = \frac{1}{6}$$

Si deve sviluppare il numeratore al $\underline{IV}$ ordine:

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + o(x^4)$$

Per sviluppare $\ln(\cos x)$ conviene iniziare dal $\cos x$:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$\ln(\cos x) = \ln\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) =$$

$$= \ln\left(1 + \underbrace{\boxed{\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)}}_{= t}\right) =$$

$$t = -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \approx x^2$$

$$o(t^n) = o((x^2)^n) = o(x^{2n}) = o(x^4)$$

$$\Longrightarrow \quad n = 2$$

$$\ln(1+t) = t - \frac{t^2}{2} + o(t^2)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2 +$$

$$+ o(x^4)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) - \frac{1}{2}\left(\left(-\frac{x^2}{2}\right)^2 + o(x^4)\right)$$

$$+ o(x^4)$$

$$t = -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \approx x^2$$

$$\frac{t}{x^2} = -\frac{1}{2} + \underset{\downarrow \atop 0}{\frac{x^2}{24}} + \underset{\downarrow \atop 0}{\frac{o(x^4)}{x^2}} \xrightarrow[x \to 0]{} -\frac{1}{2} \neq 0$$

$$\frac{t}{x^4} = -\underset{\downarrow\, x \to 0 \atop -\infty}{\frac{1}{2x^2}} + \frac{1}{24} + \underset{\downarrow \atop 0}{\frac{o(x^4)}{x^4}} \to -\infty$$

$$t = \sin x = \boxed{x} + o(x)$$

$$t \approx x$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) - \frac{1}{2}\left(\left(-\frac{x^2}{2}\right)^2 + o(x^4)\right)$$

$$+ o(x^4)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + o(x^4) =$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$$

$$\lim_{x\to 0} \frac{\ln(\cos x) + \frac{1}{2}e^{x^2} - \frac{1}{2}}{x^4} =$$

$$= \lim_{x\to 0} \frac{-\cancel{\frac{x^2}{2}} - \frac{x^4}{12} + o(x^4) + \cancel{\frac{1}{2}} + \cancel{\frac{x^2}{2}} + \frac{x^4}{4} - \cancel{\frac{1}{2}}}{x^4}$$

$$= \lim_{x\to 0} \frac{\frac{x^4}{6} + o(x^4)}{x^4} = \lim_{x\to 0} \frac{1}{6} + \frac{o(x^4)}{x^4} = \frac{1}{6}$$

$\sin(\cos x) =$

$= \sin\left(\boxed{1 - \frac{x^2}{2} + o(x^2)}\right)$ **No**

$\underset{t}{\shortparallel}$

$x \to 0 \Rightarrow t \to 1 \neq 0$

$o(x^3)$

$\sin\left(\underbrace{\boxed{x\cos x}}_{}\right)$ $= t$

$\sin t = t - \frac{t^3}{6} + o(t^3)$

$x \to 0 \Rightarrow t \to 0$

$\sin(x(1 + o(x))) =$

$\sin(x + \underline{o(x^2)}) =$

$= (x + o(x^2)) - \frac{(x + o(x^2))^3}{6} + o(x^3)$

$$\sin\left(\underbrace{\boxed{x\cos x}}_{t^4}\right) \qquad ST$$

$$t = x\cos x = x\left(1 + o(x)\right) = \boxed{x} + o(x^2)$$

$$\Rightarrow \quad t \simeq x$$

$$o(t^4) = o(x^4) = o(x^3)$$

$$n = 3$$

$$x\cos x = x\left(1 - \frac{x^2}{2} + o(x^2)\right) =$$

$$= x - \frac{x^3}{2} + o(x^3)$$

Esercizio IX

$$\lim_{x\to 0} \frac{(x+1)^{x+1} - e^{x^2} - x}{\sin x^3}$$

$$\sin x^3 = x^3 + o(x^3)$$

$\Longrightarrow$ dobbiamo sviluppare il numeratore ALMENO al III ordine:

e^{x^2}, $x^2 = t$

$x \to 0 \Longrightarrow t = x^2 \to 0$

$$o(t^n) = o(x^{2n})$$
$$\shortparallel$$
$$o(x^3) \Longrightarrow 2n = 3$$

$$n = \frac{3}{2}$$
$$\Downarrow$$
$$n = 2$$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$= 1 + x^2 + \frac{(x^2)^2}{2} + o(x^4) = \boxed{1 + x^2 + o(x^3)}$$

ATTENZIONE:

Lo sviluppo di $(1+t)^{\alpha}$ si può usare solo se l'esponente è una costante

$\alpha \in \mathbb{R} : \alpha \neq 0$

$$(1+t)^{\boxed{\alpha}} \Leftarrow$$

$$(1+n)^{\boxed{1+n}}$$ NO

$$(1+n)^{n}$$ NO

$$(1+n)^{\frac{3}{2}} = \sqrt{(1+n)^3}$$

$$(x+1)^{x+1} = e^{\ln (x+1)^{x+1}} =$$

$$= e^{(x+1)\ln(1+x)} = t$$

$$x \longrightarrow 0 \implies t = (x+1)\ln(1+x) \longrightarrow 0$$

$$t = (x+1)\ln(1+x) = (x+1)\cdot(x+o(x)) =$$

$$= x^2 + x\cdot o(x) + x + o(x) = x + o(x) \approx x$$

$$o(t^n) = o(x^n) \implies n = 3$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)$$

$$= 1 + (x+1)\ln(1+x) + \frac{((x+1)\ln(1+x))^2}{2} +$$

$$+ \frac{((x+1)\ln(1+x))^3}{6} + o(x^3)$$

$$\left(x+1\right)\ln\left(1+x\right)=\left(x+1\right)\overbrace{\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o\left(x^3\right)\right)}^{o(x^3)\ \ \text{OK}}$$

$$=x^2-\frac{x^3}{2}+\cancel{\frac{x^4}{3}}+\cancel{o\left(x^4\right)}+x-\frac{x^2}{2}+\frac{x^3}{3}+o\left(x^3\right)=$$

$$=x+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)$$

$$\left(\left(x+1\right)\ln\left(1+x\right)\right)^2=\left(x+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)\right)^2=$$

$$=\left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)^2+o\left(x^4\right)=o\left(x^3\right)$$

$$=x^2+2\cdot x\cdot\left(\frac{x^2}{2}-\cancel{\frac{x^3}{6}}\right)+\cancel{\left(\frac{x^2}{2}-\frac{x^3}{6}\right)^2}+o\left(x^3\right)$$

$$=x^2+x^3+o\left(x^3\right)$$

$$\left(\left(x+1\right)\ln\left(1+x\right)\right)^3=\left(x+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)\right)^3=$$

$$=x^3+o\left(x^3\right)$$

$$(x+1)\ln(1+x) = (x+1)\left(x - \frac{x^2}{2} + o(x^2)\right)$$

$$o(x^2)$$

$$=$$

WO

$$\left((x+1)\ln(1+x)\right)^2 = \left((x+1)\left(x - \frac{x^2}{2} + o(x^2)\right)\right)^2$$

$$= \left(x^2 - \cancel{\frac{x^3}{2}} + \cancel{x\,o(x^2)} + x - \frac{x^2}{2} + o(x^2)\right)^2 =$$

$$= \left(o(x^2) + x + \frac{x^2}{2}\right)^2 =$$

$$= \left(x + \frac{x^2}{2} + o(x^2)\right)^2 =$$

$$= \left(x + \frac{x^2}{2} + o(x^2)\right)\left(x + \frac{x^2}{2} + o(x^2)\right)$$

$$o(x^3)$$

$$= o(x^3) + \left(x + \frac{x^2}{2}\right)^2 =$$

$$= o(x^3) + x^2 + x^3$$

$$e^{(x+1)\ln(1+x)} = 1 + (x+1)\ln(1+x) + \frac{((x+1)\ln(1+x))^2}{2} + \frac{((x+1)\ln(1+x))^3}{6} + o(x^3)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3) + \frac{1}{2}\left(x^2 + x^3 + o(x^3)\right) + \frac{1}{6}\left(x^3 + o(x^3)\right) + o(x^3)$$

$$= 1 + x + x^2 + \frac{1}{2}x^3 + o(x^3)$$

$$\frac{(x+1)^{x+1} - e^{x^2} - x}{\sin x^3} =$$

$$= \frac{\cancel{1} + \cancel{x} + \cancel{x^2} + \frac{x^3}{2} - \cancel{1} - \cancel{x^2} - \cancel{x} + o(x^3)}{x^3 + o(x^3)} =$$

$$= \frac{\frac{1}{2} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} \xrightarrow[x \to 0]{} \frac{1}{2}$$

1° GIUGNO 2020 :

$$\lim_{x \to 0} \frac{\ln(1+x\cos x) - \sin x - \cos x + 1}{x^4}$$

$$\sin x = x - \frac{x^3}{6} + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$= 24$

$$\ln(1 + \underbrace{x \cos x}_{t})$$

$$x \to 0 \implies t = \overset{0}{x} \cdot \overset{1}{\cos x} \longrightarrow 0$$

$o(t^n) = ?$

$$t = x \cdot \cos x = x \cdot (1 + o(x)) \approx x$$

$$o(t^n) = o(x^n) = o(x^4)$$

$$\Longrightarrow \quad n = 4$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + o(t^4)$$

$$\ln(1 + x\cos x) =$$

$$= x \cdot \cos x - \frac{(x\cos x)^2}{2} + \frac{(x\cos x)^3}{3} -$$

$$- \frac{(x\cos x)^4}{4} + o(x^4)$$

$$x \cdot \cos x = x\left(1 - \frac{x^2}{2} + o(x^3)\right) =$$

$$= x - \frac{x^3}{2} + o(x^4)$$

$$(x \cdot \cos x)^2 \overset{?}{=} \left(x \cdot (1 + o(x)) \right)^2 =$$

$$= \left(x + o(x^2) \right)^2 =$$

$$= \left(x + o(x^2) \right) \cdot \left(x + o(x^2) \right)$$

$$x \cdot o(x^2) = o(x^3) \qquad \underline{\underline{\text{NO}}}$$

$$(x \cdot \cos x)^2 = \left(x \left(1 - \frac{x^2}{2} + o(x^2) \right) \right)^2 =$$

$$= \left(x - \frac{x^3}{2} + o(x^3) \right)^2 =$$

$$= \left(x - \frac{x^3}{2} \right)^2 + o(x^4) =$$

$$= x^2 + 2 \cdot x \left(- \frac{x^3}{2} \right) + o(x^4)$$

$$= x^2 - x^4 + o(x^4)$$

$$(x \cdot \cos x)^3 = \left(x \cdot (1 + o(x))\right)^3 =$$

$$= \left(x + o(x^2)\right)^3 =$$

$$= (x + o(x^2))(x + o(x^2))(x + o(x^2))$$

$$x^2 \cdot o(x^2) = o(x^4) \quad \checkmark$$

$$= x^3 + o(x^4)$$

$$(x \cdot \cos x)^4 = \left(x(1 + o(x))\right)^4 =$$

$$= \left(x + o(x^2)\right)^4 =$$

$$= x^4 + o(x^5) = x^4 + o(x^4)$$

$$\ln(1 + x\cos x) =$$

$$= x \cdot \cos x - \frac{(x\cos x)^2}{2} + \frac{(x\cos x)^3}{3} -$$

$$- \frac{(x\cos x)^4}{4} + o(x^4)$$

$$= x - \frac{x^3}{2} + o(x^4) - \frac{1}{2}\left(x^2 - x^4 + o(x^4)\right)$$

$$+ \frac{1}{3}\left(x^3 + o(x^4)\right) - \frac{1}{4}\left(x^4 + o(x^4)\right)$$

$$= x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{1}{4}x^4 + o(x^4)$$

$$\lim_{x \to 0} \frac{\ln(1 + x\cos x) - \sin x - \cos x + 1}{x^4} =$$

$$= \lim_{x \to 0} \frac{\cancel{x} - \cancel{\frac{x^2}{2}} - \frac{x^3}{6} + \frac{x^4}{6} + o(x^4) - \cancel{x} + \frac{x^3}{6} \cancel{- 1} + \cancel{\frac{x^2}{2}} - \frac{x^4}{24} \cancel{+ 1}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{5}{24} x^4 + o(x^4)}{x^4} =$$

$$= \lim_{x \to 0} \frac{5}{24} + \frac{o(x^4)}{x^4} = \frac{5}{24}$$

ESERCIZIO X : (eserc. teorico)

$$\lim_{n\to+\infty} \left(1+\frac{1}{n}\right)^n = e$$

$$\left(1+\frac{1}{n}\right)^n = e^{\ln\left(1+\frac{1}{n}\right)^n} =$$

$$= e^{n\ln\left(1+\frac{1}{n}\right)}$$

$$\lim_{n\to+\infty} n\ln\left(1+\frac{1}{x}\right) = ?$$

$$t = \frac{1}{n}$$

$$n\to+\infty \Longrightarrow t\longrightarrow 0$$

$$n\ln\left(1+\frac{1}{n}\right) = \frac{1}{t}\ln(1+t) = \frac{\ln(1+t)}{t}$$

$$\lim_{x \to +\infty} x \ln\left(1 + \frac{1}{x}\right) =$$

$$= \lim_{t \to 0} \frac{\ln(1+r)}{t}$$

$$\left(\ln(1+r) = t + o(t) \right)$$

$$= \lim_{r \to 0} \frac{t + o(r)}{t} = \lim_{r \to 0} \left(1 + \frac{o(r)}{t}\right) = 1$$

$$\Longrightarrow$$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to +\infty} e^{x \ln\left(1 + \frac{1}{x}\right)}$$

$$= e^1 = e$$

ESERCIZIO XI

$$\lim_{x\to 1} \frac{x^x - x}{1 - x + \ln x}$$

$x \longrightarrow 1$ (Non si possono usare gli sviluppi di Taylor per $\bar{x} = 0$!)

Facciamo il cambio di variabile

$$y = x - 1 \quad (\Longrightarrow x = y + 1)$$

$$x \longrightarrow 1 \Longrightarrow y \longrightarrow 0$$

$$\frac{x^x - x}{1 - x + \ln x} = \frac{(y+1)^{y+1} - (1+y)}{-y + \ln(1+y)}$$

$\Longrightarrow$

$$\lim_{x\to 1} \frac{x^x - x}{1 - x + \ln x} = \lim_{y\to 0} \frac{(y+1)^{y+1} - (1+y)}{-y + \ln(1+y)}$$

$$x \longrightarrow x_0$$

$$\downarrow$$

La sostituzione sarà:

$$y = x - x_0 \qquad (\rightarrow x = y + x_0)$$

$$\frac{(y+1)^{y+1} - (1+y)}{-y + \ln(1+y)}$$

Sviluppiamo il denominatore:

$$\ln(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\Longrightarrow \quad -y + \ln(1+y) = -y + y - \frac{y^2}{2} + o(y^2)$$

$$= -\frac{y^2}{2} + o(y^2)$$

Dobbiamo quindi sviluppare il numeratore al II ordine:

$$(y+1)^{y+1} = e^{\overbrace{(y+1)\ln(1+y)}^{=t}}$$

$$y \longrightarrow 0 \quad \Longrightarrow \quad t \longrightarrow 0$$

$$t = (y+1)\ln(1+y) = (y+1)\big(y + o(y)\big) \approx \ ?$$

$$t = (y+1)\ln(1+y) = (y+1)(y+o(y)) \approx \, !$$

$$\cancel{y^2} + \cancel{y\,o(y)} + y + o(y) = y + o(y)$$

$$\Longrightarrow \quad t \simeq y$$

$$o(y^2) = o(t^2)$$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$e^{\overbrace{(y+1)\ln(1+y)}^{=t}} =$$

$$= 1 + (y+1)\ln(1+y) + \frac{1}{2}\left((y+1)\ln(1+y)\right)^2 + o(y^2)$$

$$= 1 + (y+1)\left(y - \frac{y^2}{2} + o(y^2)\right) + \frac{1}{2}(y+1)^2\left(y + o(y)\right)^2$$

$$+ o(y^2)$$

$$(1+y)^{1+y} = e^{(1+y)\ln(1+y)} =$$

$$= 1 + (y+1)\left(y - \frac{y^2}{2} + o(y^2)\right) + \frac{1}{2}(y+1)^2\left(y + o(y)\right)^2$$

$$+ o(y^2)$$

$$= 1 + \left(y^2 - \cancel{\frac{y^3}{2}} + \cancel{y\,o(y^2)} + y - \frac{y^2}{2} + o(y^2)\right)$$

$$+ \frac{1}{2}\underbrace{(1 + 2y + y^2)\left(y^2 + \cancel{2y\cdot o(y)} + o(y^2)\right)}_{\left(y^2 + o(y^2) + \cancel{2y^3} + \cancel{2y\,o(y^2)} + \cancel{y^4} + \cancel{y^2 o(y^2)}\right)} + o(y^2)$$

$$= 1 + y + \frac{y^2}{2} + \frac{y^2}{2} + o(y^2) =$$

$$= \underline{1 + y + y^2 + o(y^2)}$$

$$\lim_{y\to 0} \frac{(y+1)^{y+1} - (1+y)}{-y+\ln(1+y)} =$$

$$= \lim_{y\to 0} \frac{\cancel{1} + \cancel{y} + y^2 + o(y^2) - \cancel{1} - \cancel{y}}{-\frac{y^2}{2} + o(y^2)} =$$

$$= \lim_{y\to 0} \frac{y^2 + o(y^2)}{-\frac{y^2}{2} + o(y^2)} =$$

$$= \lim_{y\to 0} \frac{1 + \frac{o(y^2)}{y^2}}{-\frac{1}{2} + \frac{o(y^2)}{y^2}} = -2$$

ESERCIZIO XII: (difficile!)

$$\lim_{x\to+\infty} x\left(\left(1+\underbrace{\frac{1}{x}}_{=y}\right)^x - e\right) = -\frac{e}{2}$$

Fare la sostituzione: $y = \frac{1}{x}$

$x \to +\infty \Longrightarrow y \to 0+$

$$x\left(\left(1+\underbrace{\frac{1}{x}}_{=y}\right)^x - e\right) = \frac{1}{y}\cdot\left(\left(1+y\right)^{\frac{1}{y}} - e\right) =$$

$$= \frac{1}{y}\cdot\left(e^{\frac{1}{y}\ln(1+y)} - e\right) =$$

$$= \frac{e}{y}\cdot\left(e^{\frac{\ln(1+y)}{y}-1} - 1\right)$$

$$= \frac{e}{y}\cdot\left(e^{\frac{\ln(1+y)-y}{y}} - 1\right)$$

$$= \frac{e}{y} \cdot \left(e^{\frac{\ln(1+y)-y}{y}} - 1 \right)$$

$$\frac{\ln(1+y)-y}{y} = \frac{y - \frac{y^2}{2} + o(y^2) - y}{y} =$$

$$= \frac{-\frac{y^2}{2} + o(y^2)}{y} = -\frac{y}{2} + \underbrace{\frac{o(y^2)}{y}}_{= o(y)}$$

$$e^{\frac{\ln(1+y)-y}{y}} = e^{-\frac{y}{2} + o(y)} =$$

$$= 1 + \left(-\frac{y}{2} + o(y)\right) + o\left(-\frac{y}{2} + \cancel{o(y)}\right)$$

$$= 1 - \frac{y}{2} + o(y)$$

$$\frac{e}{y}\cdot\left(e^{\frac{\ln(1+y)-y}{y}}-1\right)=$$

$$=\frac{e}{y}\cdot\left(\cancel{1}-\frac{y}{2}+o(y)-\cancel{1}\right)=$$

$$=\frac{-\frac{e}{2}\cdot y+o(y)}{y}=-\frac{e}{2}+\frac{o(y)}{y}$$

$$\lim_{y\to 0}\frac{e}{y}\cdot\left(e^{\frac{\ln(1+y)-y}{y}}-1\right)=$$

$$=\lim_{y\to 0}-\frac{e}{2}+\frac{o(y)}{y}=-\frac{e}{2}$$

Esercizi:

(1)

$$\lim_{x\to 0} \frac{e^x - 1 + \ln(1-x)}{\tan x - x} = -\frac{1}{2}$$

(2)

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x - \frac{3}{2}x^2}{e^{x^4} - 1} = \frac{11}{24}$$

(3)

$$\lim_{x\to 0} \frac{\ln(1 + x\cdot \arctan x) + 1 - e^{x^2}}{\sqrt{1 + \ln x^4} - 1} = -\frac{4}{3}$$

(4)

$$\lim_{x\to 0} \frac{\sin x^4 (\sin x^2 - \sin^2 x)}{1 - \cos x^4} = \frac{2}{3}$$

(5) $$\lim_{x\to 0} \frac{\sin x^4 (\sin x^2 - \sin^2 x)}{1-\cos(x^4)} = \frac{2}{3}$$

(6) * $$\lim_{x\to 0+} \frac{x^x - (\sin x)^x}{e^x - e^{-x} - 2\sin x} = \frac{1}{4}$$

(suggerimento: raccogliere preliminarmente x^x al numeratore e ricordare che $\lim_{x\to 0+} x^x = 1$ da L'Hopital)

7 *

$$\lim_{x\to+\infty} x^2 + 2x^4 \ln\left(\cos\frac{1}{x}\right) = -\frac{1}{6}$$

(suggerimento: fare il cambio di variabile $y = \frac{1}{x}$

$$x \longrightarrow +\infty \implies y \longrightarrow 0$$

e risolvere il limite

$$\lim_{y\to 0} \;\; - - - - $$)

SVOLGIMENTO:

$$\lim_{x\to 0} \frac{e^x - 1 + \ln(1-x)}{\operatorname{tg} x - x}$$

$$\operatorname{tg} x = x + \frac{x^3}{3} + o(x^3)$$

$$\operatorname{tg} x - x = \boxed{-\frac{x^3}{3}} + o(x^3)$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

$$\ln(1-x) = \ln(1 + \boxed{(-x)}) =$$

$$\left(\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3) \right)$$

$$= -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + o(x^3)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$$

$$\frac{e^x - 1 + \ln(1-x)}{\mathrm{tg}\, x - x} =$$

$$= \frac{\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + \frac{x^3}{6} + o(x^3) \cancel{-1} - \cancel{x} - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} + o(x^3)}{+ \frac{x^3}{3} + o(x^3)}$$

$$= \frac{\left(\frac{1}{6} - \frac{1}{3}\right) x^3 + o(x^3)}{+ \frac{x^3}{3} + o(x^3)} =$$

$$= \frac{-\frac{1}{6} x^3 + o(x^3)}{+\frac{1}{3} x^3 + o(x^3)} = \frac{-\frac{1}{6} + \frac{o(x^3)}{x^3}}{+\frac{1}{3} + \frac{o(x^3)}{x^3}} =$$

$$\Big\downarrow x \to 0$$

$$\frac{-\frac{1}{6}}{+\frac{1}{3}} = -\frac{1}{6} \cdot (+3) = -\frac{1}{2}$$

2

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x - \frac{3}{2}x^2}{e^{x^4} - 1} = \frac{11}{24}$$

$$e^t = 1 + t + o(t)$$

$$e^{x^4} = 1 + x^4 + o(x^4)$$

$$e^{x^4} - 1 = \boxed{x^4} + o(x^4)$$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + o(x^4)$$

$$= 1 + x^2 + \frac{x^4}{2} + o(x^4)$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$\frac{e^{x^2} - \cos x - \frac{3}{2}x^2}{e^{x^4} - 1} =$$

$$= \frac{\cancel{1} + x^2 + \frac{x^4}{2} \cancel{- 1} + \frac{x^2}{2} - \frac{x^4}{24} - \frac{3}{2}x^2 + o(x^4)}{x^4 + o(x^4)}$$

$$= \frac{\left(\frac{1}{2} - \frac{1}{24}\right)x^4 + o(x^4)}{x^4 + o(x^4)}$$

$$= \frac{\frac{11}{24}x^4 + o(x^4)}{x^4 + o(x^4)} = \frac{\frac{11}{24} + \frac{o(x^4)}{x^4}}{1 + \frac{o(x^4)}{x^4}}$$

$$\xrightarrow{x \to 0} \frac{11}{24}$$

$$\lim_{x \to x_0} f(x) = \ell$$

$$f : A \to \mathbb{R}$$

$$x_0 \in \mathcal{D}(A)$$

$$\forall \varepsilon > 0 \, , \; \exists \delta = \delta(x_0, \varepsilon) > 0 :$$

$$\forall x \in \mathbb{D}(f) : \quad \underline{0 <} \, |x - x_0| < \delta$$

$$\Longrightarrow \; |f(x) - \ell| < \varepsilon$$

$$f(x) = \frac{x^2+2x-4}{x+2} \cdot e^x$$

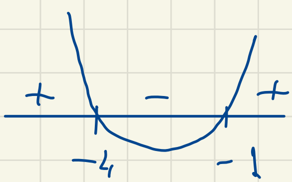
$$D(f) = \mathbb{R} \setminus \{-2\}$$

$$f' = \frac{e^x}{(x+2)^2} \cdot (x^3+5x^2+4x)$$

$$x\,(x^2+5x+4)$$

$$\Delta = 25-16 = 9 > 0$$

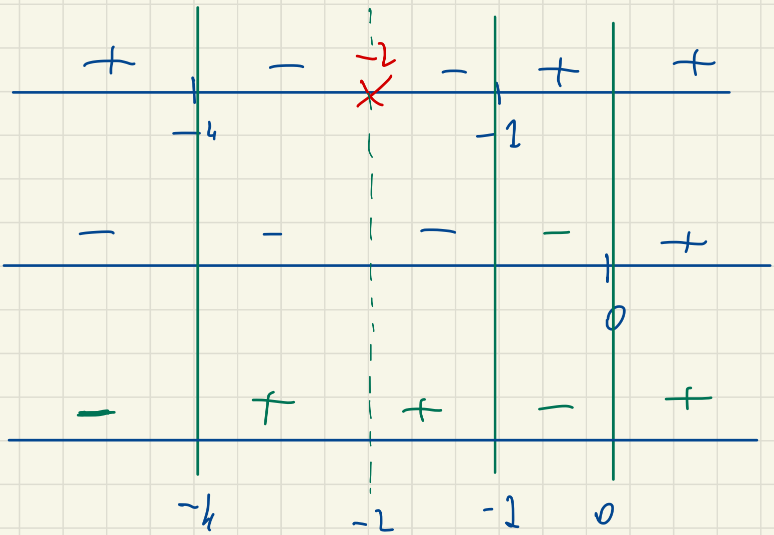
$$x_{1,2} = \frac{-5 \pm 3}{2} \begin{cases} -4 \\ -1 \end{cases}$$



$x(x^2+5x+4)$

x^2+5x+4

x

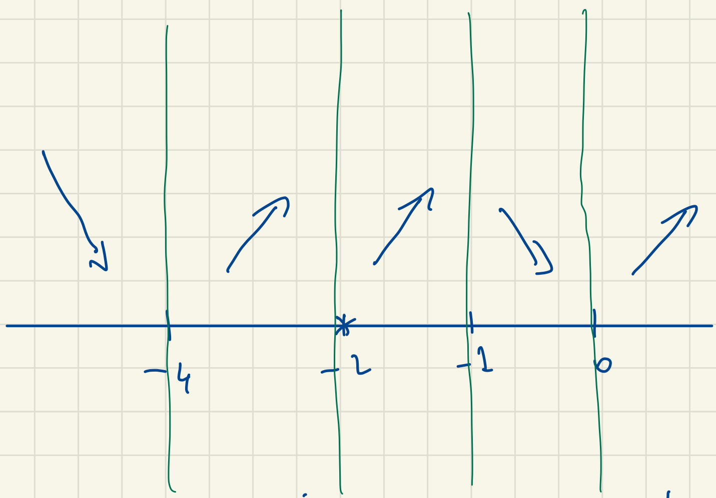


$x(x^2+5x+4)$

f'

hanno lo stesso segno

f



$x=-4$ p. di min. relativo $f(-4)=!$

$x=-1$ p. di max relativo $f(-1)=!$

$x = 0$ $\quad$ p. di min. relativo $\quad f(0) = 1$

④ $$\lim_{x\to 0} \frac{\sin x^4 (\sin x^2 - \sin^2 x)}{1-\cos x^4} = \frac{2}{3}$$

$$1-\cos x^4 = 1-\left(1-\frac{(x^4)^2}{2}+o(x^8)\right)$$

$$= +\frac{x^8}{2}+o(x^8)$$

$$\sin t = t + o(t^2)$$

$$\sin x^4 = x^4 + o(x^8)$$

$$\sin x^2 = x^2 + o(x^4)$$

$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3)\right)^2 =$$

$$= x^2 + 2\cdot x \cdot \left(-\frac{x^3}{6}\right) + o(x^4)$$

$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$\sin x^4 \left(\sin x^2 - \sin^2 x\right) =$$

$$= \left(x^4 + o(x^8)\right)\left(\cancel{x^2} + o(x^4) - \cancel{x^2} + \frac{x^4}{3} + o(x^4)\right)$$

$$= \frac{x^8}{3} + o(x^8)$$

$$\lim_{x \to 0} \frac{\sin x^4 \left(\sin x^2 - \sin^2 x\right)}{1 - \cos x^4} =$$

$$= \lim_{x \to 0} \frac{\frac{x^8}{3} + o(x^8)}{\frac{x^8}{2} + o(x^8)} =$$

$$= \lim_{x \to 0} \frac{\frac{1}{3} + \frac{o(x^8)}{x^8}}{\frac{1}{2} + \frac{o(x^8)}{x^8}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$