# 6. Dicembre. 2021

$$\lim_{x \to 0} \frac{\sin x^3 - \sin^3 x}{\sin x^5}$$

$$\sin t = t + o(t)$$

- $\sin x^5 = x^5 + o(x^5) \Rightarrow m = 5$

Dobbiamo sviluppare il numeratore al V ordine:

- $\sin \overbrace{x^3}^{= t}$

$$o(t^n) = o\left((x^3)^n\right) = o(x^{3n})$$

$$3n = 5 \Rightarrow \text{"} n = \frac{5}{3} \text{"} \Rightarrow n = 2$$

$$\sin t = t + o(t^2)$$

$$= x^3 + o(x^6) = x^3 + o(x^5)$$

$$(\sin x)^3 = (x + o(x^2))^3 =$$

$$= (x + o(x^2))(x + o(x^2))(x + o(x^2))$$

$x^2 o(x^2) = o(x^4)$ <u>NO</u>

$$(\sin x)^3 = \left(x - \frac{x^3}{6} + o(x^3)\right)^3 =$$

$$= \left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$o(x^3) \cdot x^2 = o(x^5)$

$o(x^3) \cdot x \cdot \left(-\frac{x^3}{6}\right) = o(x^7) = o(x^5)$

...

$$= \left(x - \frac{x^3}{6}\right)^3 + o(x^5)$$

$$\sin^3 x = \left(x - \frac{x^3}{6}\right)^3 + o(x^5)$$

$$= \left(x^3 + 3 \cdot x^2 \cdot \left(-\frac{x^3}{6}\right) + \cancel{3 \cdot x \left(-\frac{x^3}{6}\right)^2} + \cancel{\left(-\frac{x^3}{6}\right)^3}\right)$$

$$+ o(x^5)$$

$$= x^3 - \frac{x^5}{2} + o(x^5)$$

Sostituendo gli sviluppi calcolati:

$$\frac{\sin x^3 - \sin^3 x}{\sin x^5} = \frac{x^3 + o(x^5) - \left(x^3 - \frac{x^5}{2} + o(x^5)\right)}{x^5 + o(x^5)}$$

$$= \frac{\cancel{x^3} - \cancel{x^3} + \frac{x^5}{2} + o(x^5)}{x^5 + o(x^5)} =$$

$$= \frac{\frac{1}{2} + \frac{o(x^5)}{x^5}}{1 + \frac{o(x^5)}{x^5}} \xrightarrow[x \to 0]{} \frac{1}{2}$$

## Esercizio I (sui limiti):

$$\lim_{x \to 0} \frac{e^x - 1 + \ln(1-x)}{x^3} = ?$$

Si deve sviluppare il numeratore (almeno) al III ordine:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3)$$

$$\ln\big(1 + \underbrace{(-x)}_{= t}\big) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} + o(x^3)$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^3)$$

$\Longrightarrow$

$$e^x - 1 + \ln(1-x) = \cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + \frac{x^3}{3!} + o(x^3) \cancel{-1} -$$

$$\cancel{-x} - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} + o(x^3) =$$

$$= -\frac{1}{6} x^3 + o(x^3)$$

$$\lim_{x \to 0} \frac{e^x - 1 + \ln(1-x)}{x^3} =$$

$$= \lim_{x \to 0} \frac{-\frac{1}{6}x^3 + o(x^3)}{x^3} =$$

$$= \lim_{x \to 0} -\frac{1}{6} + \frac{o(x^3)}{x^3} = -\frac{1}{6}$$

## Esercizio II :

$$\lim_{x \to 0} \frac{\ln(1+\sin x) - \ln(1+x) + \frac{1}{6}\sin^3 x}{\sin(x^4)}$$

$\sin(x^4) = ?$

$\sin t = t + o(t)$

$\hookrightarrow \quad \sin(x^4) = x^4 + o(x^4)$

Bisogna sviluppare il numeratore al IV ordine $(o(x^4))$ :

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)}$$

$$\sin^3 x = \left(x + o(x)\right)^3 = \qquad \boxed{\text{NO}}$$

$$= \left(x + o(x)\right)\cdot\left(x + o(x)\right)\cdot\left(x + o(x)\right)$$

$$x^2 \cdot o(x) = o(x^3) \qquad \boxed{\text{NO}}$$

$$\sin^3 x = \left(x + 0\cdot x^2 + o(x^2)\right)^3 =$$

$$= \left(x + o(x^2)\right)^3 =$$

$$= \left(x + o(x^2)\right)\cdot\left(x + o(x^2)\right)\cdot\left(x + o(x^2)\right)$$

$$x^2 \cdot o(x^2) = o(x^4) \qquad \text{OK}$$

$$= x^3 + o(x^4)$$

$$\ln(1+\sin x) = \quad ? \quad + o(x^4)$$

$$\sin x = t$$

$$x \longrightarrow 0 \implies t \longrightarrow 0$$

$$\ln(1+t) = \sum_{j=1}^{n} (-1)^{j-1} \frac{t^j}{j} + o(t^n)$$

$$t = \sin x = x + o(x) \simeq x$$

$$o(t^n) = o(\sin^n x) = o(x^n)$$

$$\implies o(t^n) = o(x^n) = o(x^4)$$

$$\implies n = 4$$

$$\ln(1+t) = \sum_{j=1}^{4} (-1)^{j-1} \frac{t^j}{j} + o(t^4)$$

$$\ln(1+t) = \sum_{j=1}^{4} (-1)^{j-1} \frac{t^j}{j} + o(t^4)$$

$$= t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + o(t^4)$$

$$\ln(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + o(x^4)$$

serve la prima
<u>fine</u> dell'errore

$$\boxed{\sin x = x - \frac{x^3}{6} + o(x^4)}$$

$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3)\right)^2 =$$

$$= \left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$x \cdot o(x^3) = o(x^4)$ sì

$-\frac{x^3}{6} \cdot o(x^3) = o(x^6)$ X

$o(x^3) \cdot o(x^3) = o(x^6)$ X

$$\underline{\underline{\sin^2 x}} = \left(x - \frac{x^3}{6}\right)^2 + o(x^4)$$

$$= x^2 + 2 \cdot x \cdot \left(-\frac{x^3}{6}\right) + \cancel{\left(-\frac{x^3}{6}\right)^2} + o(x^4)$$

$$= \boxed{x^2 - \frac{x^4}{3} + o(x^4)}$$

$$\sin^3 x = \left(x - \frac{x^3}{6} + o(x^3)\right)^3 =$$

$$= \left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$x^2 \cdot o(x^3) = o(x^5) = o(x^4)$$

$$= \left(x - \frac{x^3}{6}\right)^3 + o(x^4) =$$

$$= x^3 + \cancel{3 \cdot x^2 \cdot \left(-\frac{x^3}{6}\right)} + \dots + o(x^4)$$

$$\sin^3 x = \left(\underline{x + o(x^2)}\right)^3 =$$

$$= (x+o(x^2)) \cdot (x+o(x^2)) \cdot (x+o(x^2)) =$$

$$x^2 \cdot o(x^2) = o(x^4) \ \underline{\text{OK}}$$

$$= x^3 + o(x^4) \leftarrow$$

$$\sin x = (x + o(x))^3$$

$$\uparrow$$

$$x^2 \cdot o(x) = o(x^3) \ \underline{\underline{\text{NO}}}$$

$$\underline{\underline{\sin^3 x}} = \boxed{x^3 + o(x^4)}$$

<u>Nota</u>: per fare il calcolo di $\sin^3 x$ con un $o(x^4)$ si può sviluppare il $\sin x$ solo al II ordine:

$$\sin^3 x = \left(x + o(x^2)\right)^3 =$$

$$= \left(x + o(x^2)\right)\left(x + o(x^2)\right)\left(x + o(x^2)\right)$$

$$o(x^2) \cdot x^2 = o(x^4)$$

$$= x^3 + o(x^4)$$

$$\sin^4 x = (x + o(x))^4 =$$

$$= \boxed{x^4 + o(x^4)}$$

$$\ln(1+\sin x) = \sin x - \frac{\sin^2 x}{2} + \frac{\sin^3 x}{3} - \frac{\sin^4 x}{4} + o(x^4)$$

$$= x - \frac{x^3}{6} + o(x^4) - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + o(x^4)\right) + \frac{1}{3}\left(x^3 + o(x^4)\right) - \frac{1}{4}\left(x^4 + o(x^4)\right) + o(x^4)$$

$$= x - \frac{x^3}{6} - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$= \boxed{x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{1}{12}x^4 + o(x^4)}$$

Riscriviamo gli sviluppi calcolati:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)$$

$$\sin^3 x = x^3 + o(x^4)$$

$$\ln(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + o(x^4)$$

$$\sin(x^4) = x^4 + o(x^4)$$

$$\Longrightarrow \lim_{x\to 0} \frac{\ln(1+\sin x) - \ln(1+x) + \frac{1}{6}\sin^3 x}{\sin(x^4)} =$$

$$= \frac{\cancel{x} - \cancel{\frac{x^2}{2}} + \frac{x^3}{6} - \frac{x^4}{12} - \cancel{x} + \cancel{\frac{x^2}{2}} - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^3}{6} + o(x^4)}{x^4 + o(x^4)}$$

$$= \frac{-\frac{x^4}{12} + \frac{x^4}{4} + o(x^4)}{x^4 + o(x^4)} = \frac{\frac{-1+3}{12}x^4 + o(x^4)}{x^4 + o(x^4)}$$

$$\frac{\frac{-1+3}{12}x^4+o(x^4)}{x^4+o(x^4)} = \frac{\frac{1}{6}x^4+o(x^4)}{x^4+o(x^4)} =$$

$$\lim_{x\to 0} \frac{\ln(1+\sin x)-\ln(1+x)+\frac{1}{6}\sin^3 x}{\sin(x^4)} =$$

$$= \lim_{x\to 0} \frac{\cancel{x^4}}{\cancel{x^4}} \cdot \frac{\frac{1}{6}+\frac{o(x^4)}{x^4}}{1+\frac{o(x^4)}{x^4}} = \frac{1}{6}$$

## Esercizio III :

$$\lim_{x \to 0} \frac{\sqrt[4]{1-4x^2+x^4} - 1 + x^2 + x^6}{x^4}$$

ATTENZIONE a sviluppare senza tenere conto degli errori_

Procedimento ERRATO:

$$\sqrt[4]{1+t} = (1+t)^{\frac{1}{4}} \cong 1 + \frac{1}{4} t$$

$$\sqrt[4]{1+(-4x^2+x^4)} \cong 1 + \frac{1}{4}(-4x^2+x^4) =$$
$$= 1 - x^2 + \frac{1}{4} x^4$$

$$\frac{\sqrt[4]{1-4x^2+x^4} - 1 + x^2 + x^6}{x^4} = \frac{\cancel{1} - \cancel{x^2} + \frac{1}{4}x^4 - \cancel{1} + \cancel{x^2} + x^6}{x^4}$$

$$= \frac{1}{4} + x^2 \xrightarrow[x \to 0]{} \frac{1}{4}$$

NO!!

Procedimento corretto:

$$\lim_{x\to 0} \frac{\sqrt[4]{1-4x^2+x^4} - 1 + x^2 + x^6}{x^4}$$

$\boxed{n=4}$

$$\sqrt[4]{1+\underbrace{(-4x^2+x^4)}_{t}} = (1+t)^{\frac{1}{4}}$$

$t = -4x^2 + x^4 \approx x^2$

$$\Rightarrow \quad o(t^n) = o((x^2)^n) = o(x^{2n}) = o(x^4)$$

$2n = 4$

$n = 2$

$o(t^2)$

$$\sqrt[4]{1+t} = (1+t)^{\frac{1}{4}} = 1 + \frac{1}{4}t + \binom{\frac{1}{4}}{2} t^2 + o(t^2) =$$

$$= 1 + \frac{1}{4}t + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2} t^2 + o(t^2)$$

$$(1+t)^{\alpha} = \sum_{j=0}^{n} \binom{\alpha}{j} \cdot t^{j} + o(t^{n})$$

$n = 2$ :

$$(1+r)^{\alpha} = 1 + \binom{\alpha}{1} t^{1} + \binom{\alpha}{2} t^{2} +$$

$$+ o(r^{2}) =$$

$$= 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^{2} + o(r^{2})$$

$$(1+r)^{\frac{1}{4}} = 1 + \frac{1}{4} t + \frac{\frac{1}{4}\left(\frac{1}{4}-1\right)}{2} t^{2} + o(r^{2})$$

$$\sqrt[4]{1+t} = 1 + \frac{1}{4}t - \frac{3}{32}t^2 + \overset{o(x^4)}{\overset{\shortparallel}{o(t^2)}}$$

$$\sqrt[4]{1+(-4x^2+x^4)} = 1 + \frac{1}{4}(-4x^2+x^4) -$$

$$- \frac{3}{32}(-4x^2+x^4)^2 + o(x^4)$$

$$= 1 - x^2 + \frac{1}{4}x^4 - \frac{3}{2}x^4 + o(x^4)$$

$$= 1 - x^2 - \frac{5}{4}x^4 + o(x^4)$$

$$\frac{\sqrt[4]{1-4x^2+x^4} - 1 + x^2 + x^6}{x^4} =$$

$$= \frac{\cancel{1} - \cancel{x^2} - \frac{5}{4}x^4 + o(x^4) \cancel{-1} + \cancel{x^2} + x^6}{x^4} =$$

$$= -\frac{5}{4} + \frac{o(x^4)}{x^4} + x^2 \xrightarrow[x\to 0]{} -\frac{5}{4}$$

$\Longrightarrow$

$$\lim_{x \to 0} \frac{\sqrt[4]{1-4x^2+x^4} - 1 + x^2 + x^6}{x^4} = -\frac{5}{4}$$

# Esercizio IV:

$$\lim_{x \to 0} \frac{5^{1+\operatorname{tg}^2 x} - 5}{1-\cos x}$$

Metodo di risoluzione:

- si inizia con lo sviluppare la funzione più semplice al numeratore o al denominatore.

In questo caso il denominatore:

$$\underline{1-\cos x} = 1-\left(1-\frac{x^2}{2}+o(x^2)\right) =$$

$$= \boxed{\frac{x^2}{2}+o(x^2)}$$

$\Longrightarrow$ dobbiamo sviluppare il numeratore ALMENO al II ordine

## Attenzione:

$$5^{1+\operatorname{tg}^2 x} = e^{\ln 5^{1+\operatorname{tg}^2 x}} =$$

$$= e^{(1+\operatorname{tg}^2 x)\cdot \ln 5}$$

Non si può scegliere $t = (1+\operatorname{tg}^2 x)\cdot \ln 5$

poiché:

$$x \to 0 \longrightarrow t \longrightarrow \ln 5 \neq 0$$

Si procede come segue:

$$5^{1+\operatorname{tg}^2 x} - 5 = 5\left(5^{\operatorname{tg}^2 x} - 1\right)$$

$$5^{\operatorname{tg}^2 x} = e^{\ln 5^{\operatorname{tg}^2 x}} = e^{\overbrace{(\ln 5)\operatorname{tg}^2 x}^{=t}}$$

$$x \longrightarrow 0 \implies t = \operatorname{tg}^2 x \longrightarrow 0$$

$$\operatorname{tg} x = x + o(x) \implies \operatorname{tg}^2 x \sim x^2$$

$$o(t^n) = o(x^{2n}) \implies n = 1$$

$$5^{\mathrm{tg}^2 x} = e^{\ln 5^{\mathrm{tg}^2 x}} = e^{\overbrace{(\ln 5)\,\mathrm{tg}^2 x}^{=t}}$$

$$= 1 + (\ln 5)\,\mathrm{tg}^2 x + o(x^2)$$

$$= 1 + (\ln 5)\,\big(x + o(x)\big)^2 + o(x^2)$$

$$= 1 + (\ln 5)\,\big(x^2 + \underbrace{2x\,o(x) + o(x)^2}_{o(x^2)}\big) + o(x^2)$$

$$= 1 + (\ln 5)\,x^2 + o(x^2)$$

Inserendo tutti gli sviluppi calcolati:

$$\frac{5^{1+\operatorname{tg}^2 x} - 5}{1-\cos x} = \frac{5\left(1 + (\ln 5)\, x^2 + o(x^2) - 1\right)}{\frac{x^2}{2} + o(x^2)}$$

$$= \frac{5(\ln 5)\, x^2 + o(x^2)}{\frac{x^2}{2} + o(x^2)} =$$

$$= \frac{5(\ln 5) + \frac{o(x^2)}{x^2}}{\frac{1}{2} + \frac{o(x^2)}{x^2}} \xrightarrow[x \to 0]{} 10\,(\ln 5)$$

# ESERCIZIO V :

$$\lim_{x\to 0} \frac{\ln(1+x^5)}{\sin(\sin x) - \sin x + \frac{x^3}{6}}$$

Iniziamo dal numeratore:

$$\ln(1+x^5) = x^5 + o(x^5)$$

$\Longrightarrow$ si deve sviluppare il denominatore al V ordine: $(o(x^5))$

$$\sin(\underbrace{\sin x}_{=t})$$

$x \to 0 \Longrightarrow t \to 0$ OK

$t = \sin x \simeq x \Longrightarrow o(t^n) = o(x^n)$

$\Longrightarrow n = 5$

$$\sin t = \sum_{k=0}^{2} \frac{(-1)^k \cdot t^{2k+1}}{(2k+1)!} + o(t^5) =$$

$$= t - \frac{t^3}{6} + \frac{t^5}{5!} + o(t^5)$$

lo teniamo così!

$$\sin(\sin x) = \sin x - \frac{\sin^3 x}{6} + \frac{\sin^5 x}{5!} + o(x^5)$$

Vogliamo sviluppare $\sin^3 x$; quale sviluppo dobbiamo usare per $\sin x$?

Per tentativi:

$$\sin^3 x = \left(x + o(x^2)\right)^3$$

$$= \left(x+o(x^2)\right)\left(x+o(x^2)\right)\left(x+o(x^2)\right)$$

$x^2 \cdot o(x^2) = o(x^4)$ **NON** è un $o(x^5)$

$$\sin^3 x =$$

$$= \left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right)$$

$$x^2 \cdot o(x^3) = o(x^5)$$

$$= \left(x - \frac{x^3}{6}\right)^3 + o(x^5) =$$

$$= x^3 + 3 \cdot x^2 \cdot \left(-\frac{x^3}{6}\right) + \cancel{\dots} + o(x^5) =$$

$$= x^3 - \frac{x^5}{2} + o(x^5)$$

Sviluppiamo $\sin^5 x$ :

$$\sin^5 x = (x + o(x))^5 =$$

$$= (x+o(x))(x+o(x))(x+o(x))(x+o(x))(x+o(x)) =$$

$$x^4 \cdot o(x) = o(x^5)$$

$$= x^5 + o(x^5)$$

Sostituendo gli sviluppi troviamo:

$$\sin(\sin x) = \sin x - \frac{\sin^3 x}{6} + $$

$$+ \frac{\sin^5 x}{5!} + o(x^5)$$

$$= \sin x - \frac{1}{6}\left(x^3 - \frac{x^5}{2} + o(x^5)\right) +$$

$$+ \frac{1}{120}\left(x^5 + o(x^5)\right) =$$

$$= \sin x - \frac{x^3}{6} + \frac{x^5}{12} + \frac{x^5}{120} =$$

$$= \sin x - \frac{x^3}{6} + \frac{11}{120} x^5 + o(x^5)$$

$\Longrightarrow$

$$\sin(\sin x) - \sin x = -\frac{x^3}{6} + \frac{11}{120} x^5 + o(x^5)$$

$$\lim_{x\to 0} \frac{\ln(1+x^5)}{\sin(\sin x) - \sin x + \frac{x^3}{6}} =$$

$$= \lim_{x\to 0} \frac{x^5 + o(x^5)}{\cancel{-\frac{x^3}{6}} + \frac{1}{120}x^5 + o(x^5) + \cancel{\frac{x^3}{6}}} =$$

$$= \lim_{x\to 0} \frac{1 + \frac{o(x^5)}{x^5}}{\frac{1}{120} + \frac{o(x^5)}{x^5}} = \frac{120}{1}$$

# ESERCIZIO VI

$$\lim_{x\to 0} \left( \frac{1}{x^2} - \frac{1}{x \cdot \sin x} \right) \qquad [+\infty - \infty]$$

$$\frac{1}{x^2} - \frac{1}{x \cdot \sin x} = \frac{\sin x - x}{x^2 \cdot \sin x}$$

$$x^2 \cdot \sin x = x^2 \left( x + o(x) \right) =$$
$$= x^3 + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\frac{1}{x^2} - \frac{1}{x \cdot \sin x} = \frac{-\frac{x^3}{6} + o(x^3)}{x^3 + o(x^3)} = \frac{-\frac{1}{6} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}}$$

$$\Big\downarrow x \to 0$$

$$-\frac{1}{6}$$

# ESERCIZIO VII :

$$\lim_{x\to 0} (\cos x)^{-\frac{1}{x^2}} \qquad [1^{-\infty}]$$

$$(\cos x)^{-\frac{1}{x^2}} = e^{\ln(\cos x)^{-\frac{1}{x^2}}} =$$

$$= e^{-\frac{1}{x^2}\ln \cos x} = e^{-\frac{\ln \cos x}{x^2}}$$

$$\lim_{x\to 0} -\frac{\ln \cos x}{x^2} = ?$$

si deve sviluppare il $\ln(\cos x)$ al II ordine :

$$\ln \cos x = \ln\Big(1 + \underbrace{\boxed{\left(-\frac{x^2}{2} + o(x^2)\right)}}_{=\,t}\Big)$$

$$t \approx x^2 \Rightarrow o(t) = o(x^2)$$

$$\ln(1+t) = t + o(t) = -\frac{x^2}{2} + o(x^2)$$

$$\ln \cos x = -\frac{x^2}{2} + o(x^2)$$

$$\lim_{x \to 0} -\frac{\ln \cos x}{x^2} = \lim_{x \to 0} -\frac{-\frac{x^2}{2} + o(x^2)}{x^2} =$$

$$= \lim_{x \to 0} +\frac{1}{2} - \frac{o(x^2)}{x^2} = -\frac{1}{2}$$

$\Rightarrow$

$$\lim_{x \to 0} (\cos x)^{-\frac{1}{x^2}} =$$

$$= \lim_{x \to 0} e^{-\frac{\ln \cos x}{x^2}} = e^{+\frac{1}{2}} = \sqrt{e}$$

# ESERCIZIO VII :

$$\lim_{x \to 0} (\cos x)^{-\frac{1}{x^2}} \qquad [1^{-\infty}]$$

$$(\cos x)^{-\frac{1}{x^2}} = e^{\ln (\cos x)^{-\frac{1}{x^2}}} =$$

$$= e^{-\frac{1}{x^2} \ln \cos x} = e^{-\frac{\ln \cos x}{x^2}}$$

$$\lim_{x \to 0} -\frac{\ln \cos x}{x^2} = ?$$

Si deve sviluppare il $\ln (\cos x)$ al II ordine :

$$\ln \cos x = \ln \Big(1 + \underbrace{\boxed{\left(-\frac{x^2}{2} + o(x^2)\right)}}_{= t}\Big)$$

$$t \approx x^2 \Rightarrow o(t) = o(x^2)$$

$$= \ln (1+t) = t + o(t) = -\frac{x^2}{2} + o(x^2)$$

$$\ln(1+r) = t + o(t)$$

$$\ln\left(1+\left(-\frac{x^2}{2}+o(x^2)\right)\right) =$$

$$= -\frac{x^2}{2} + \underline{o(x^2)} + \underbrace{o\left(-\frac{x^2}{2}+o(x^2)\right)}_{u}$$

$$o\left(-\frac{x^2}{2}\right) = o(x^2)$$

$$= -\frac{x^2}{2} + o(x^2)$$

$$\ln(1+r) = r - \frac{r^2}{2} + \frac{r^3}{3} + o(r^3)$$

$$\ln(1+r) = t + o(r)$$

$$\ln \cos x = -\frac{x^2}{2} + o(x^2)$$

$$\lim_{x \to 0} - \frac{\ln \cos x}{x^2} = \lim_{x \to 0} - \frac{-\frac{x^2}{2} + o(x^2)}{x^2} =$$

$$= \lim_{x \to 0} + \frac{1}{2} - \frac{o(x^2)}{x^2} = \frac{1}{2}$$

$\Longrightarrow$

$$\lim_{x \to 0} (\cos x)^{-\frac{1}{x^2}} =$$

$$= \lim_{x \to 0} e^{-\frac{\ln \cos x}{x^2}} = e^{+\frac{1}{2}} = \sqrt{e}$$

## Esercizio VIII :

$$\lim_{x \to 0} \frac{\ln(\cos x) + \frac{1}{2} e^{x^2} - \frac{1}{2}}{x^4} = \frac{1}{6}$$

Si deve sviluppare il numeratore al IV ordine :

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2} + o(x^4)$$

Per sviluppare $\ln(\cos x)$ conviene iniziare dal $\cos x$ :

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$$

$$\ln(\cos x) = \ln\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) =$$

$$= \ln\Bigg(1 + \underbrace{\boxed{\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)}}_{\displaystyle = t}\Bigg) =$$

$$t = -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \approx x^2$$

$$o(t^n) = o\left((x^2)^n\right) = o(x^{2n}) = o(x^4)$$

$$\Longrightarrow \quad n = 2$$

$$\ln(1+t) = t - \frac{t^2}{2} + o(t^2)$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + o(x^4)\right)^2 +$$

$$+ o(x^4)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) - \frac{1}{2}\left(\left(-\frac{x^2}{2}\right)^2 + o(x^4)\right)$$

$$+ o(x^4)$$

$$t = -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) \approx x^2$$

$$\frac{t}{x^2} = -\frac{1}{2} + \underset{\downarrow \atop 0}{\frac{x^2}{24}} + \underset{\downarrow \atop 0}{\frac{o(x^4)}{x^2}} \xrightarrow[x \to 0]{} -\frac{1}{2} \neq 0$$

$$\frac{t}{x^4} = -\underset{\downarrow\, x \to 0 \atop -\infty}{\frac{1}{2x^2}} + \frac{1}{24} + \underset{\downarrow \atop 0}{\frac{o(x^4)}{x^4}} \longrightarrow -\infty$$

$$t = \sin x = \boxed{x} + o(x)$$

$$r \approx x$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} + o(x^4) - \frac{1}{2}\left(\left(-\frac{x^2}{2}\right)^2 + o(x^4)\right)$$

$$+ o(x^4)$$

$$= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^4}{8} + o(x^4) =$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} + o(x^4)$$

$$\lim_{x \to 0} \frac{\ln(\cos x) + \frac{1}{2} e^{x^2} - \frac{1}{2}}{x^4} =$$

$$= \lim_{x \to 0} \frac{-\cancel{\frac{x^2}{2}} - \frac{x^4}{12} + o(x^4) + \cancel{\frac{1}{2}} + \cancel{\frac{x^2}{2}} + \frac{x^4}{4} - \cancel{\frac{1}{2}}}{x^4}$$

$$= \lim_{x \to 0} \frac{\frac{x^4}{6} + o(x^4)}{x^4} = \lim_{x \to 0} \frac{1}{6} + \frac{o(x^4)}{x^4} = \frac{1}{6}$$

$\sin(\cos x) =$

**No**

$= \sin\left(\boxed{1 - \frac{x^2}{2} + o(x^2)}\right)$

$\shortparallel$

$t$

$x \to 0 \implies t \to 1 \neq 0$

$o(x^3)$

$\sin\left(\boxed{x \cos x}\right)$ with $\boxed{x \cos x} = t$

$\sin t = t - \frac{t^3}{6} + o(t^3)$

$\uparrow$

$x \to 0 \implies t \to 0$

$\sin\left(x\left(1 + o(x)\right)\right) =$

$\sin\left(x + \underline{o(x^2)}\right) =$

$= \left(x + o(x^2)\right) - \frac{(x + o(x^2))^3}{6} + o(x^3)$

$\uparrow$

$$\sin\left(\left|\underbrace{\boxed{x\cos x}}_{t^n}\right|\right) \qquad \sqrt{\ }$$

$$t = x\cos x = x(1+o(x)) = \boxed{x} + o(x^2)$$

$$\Rightarrow \quad t \simeq x$$

$$o(t^n) = o(x^n) = o(x^3)$$

$$n = 3$$

$$x\cos x = x\left(1 - \frac{x^2}{2} + o(x^2)\right) =$$

$$= x - \frac{x^3}{2} + \underline{\underline{o(x^3)}}$$

# Esercizio IX

$$\lim_{x\to 0} \frac{(x+1)^{x+1} - e^{x^2} - x}{\sin x^3}$$

$$\sin x^3 = x^3 + o(x^3)$$

$\Longrightarrow$ dobbiamo sviluppare il numeratore ALMENO al III ordine:

$e^{x^2}$, $x^2 = t$

$x \to 0 \Longrightarrow t = x^2 \to 0$

$$o(t^n) = o(x^{2n})$$
$$\shortparallel$$
$$o(x^3) \Longrightarrow 2n = 3$$

$$n = \frac{3}{2}$$
$$\Downarrow$$
$$n = 2$$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$$= 1 + x^2 + \frac{(x^2)^2}{2} + o(x^4) = \boxed{1 + x^2 + o(x^3)}$$

# ATTENZIONE:

Lo sviluppo di $(1+t)^{\alpha}$ si può usare SOLO se l'esponente è una costante

$\alpha \in \mathbb{R} : \alpha \neq 0$

$$(1+t)^{\boxed{\alpha}}$$

$$(1+x)^{\boxed{1+x}} \qquad \text{NO}$$

$$(1+x)^{x} \qquad \text{NO}$$

$$(1+x)^{\frac{3}{2}} = \sqrt{(1+x)^{3}} \qquad ✓$$

$$(x+1)^{x+1} = e^{\ln (x+1)^{x+1}} =$$

$$= e^{\overbrace{(x+1)\ln(1+x)}^{=t}}$$

$$x \longrightarrow 0 \implies t = (x+1)\ln(1+x) \longrightarrow 0$$

$$t = (x+1)\ln(1+x) = (x+1)\cdot(x+o(x)) =$$

$$= x^2 + x\cdot o(x) + x + o(x) = x + o(x) \approx x$$

$$o(t^n) = o(x^n) \implies n = 3$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)$$

$$= 1 + (x+1)\ln(1+x) + \frac{((x+1)\ln(1+x))^2}{2} +$$

$$+ \frac{((x+1)\ln(1+x))^3}{6} + o(x^3)$$

$o(x^3)$ OK

$$(x+1)\ln(1+x) = (x+1)\left(x-\frac{x^2}{2}+\frac{x^3}{3}+o(x^3)\right)$$

$$= x^2 - \frac{x^3}{2} + \cancel{\frac{x^4}{3}} + \cancel{o(x^4)} + x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3) =$$

$$= x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3)$$

$$\left((x+1)\ln(1+x)\right)^2 = \left(x+\frac{x^2}{2}-\frac{x^3}{6}+o(x^3)\right)^2 =$$

$$= \left(x+\frac{x^2}{2}-\frac{x^3}{6}\right)^2 + o(x^4) = o(x^3)$$

$$= x^2 + 2\cdot x\cdot\left(\frac{x^2}{2}-\cancel{\frac{x^3}{6}}\right) + \left(\cancel{\frac{x^2}{2}-\frac{x^3}{6}}\right)^2 + o(x^3)$$

$$= x^2 + x^3 + o(x^3)$$

$$\left((x+1)\ln(1+x)\right)^3 = \left(x+\frac{x^2}{2}-\frac{x^3}{6}+o(x^3)\right)^3 =$$

$$= x^3 + o(x^3)$$

$$(x+1)\ln(1+x) = (x+1)\left(x - \frac{x^2}{2} + o(x^2)\right)$$

$$o(x^2)$$

$$=$$

W0

$$\left((x+1)\ln(1+x)\right)^2 = \left((x+1)\left(x - \frac{x^2}{2} + o(x^2)\right)\right)^2$$

$$= \left(x^2 - \cancel{\frac{x^3}{2}} + \cancel{x\,o(x^2)} + x - \frac{x^2}{2} + o(x^2)\right)^2 =$$

$$= \left(o(x^2) + x + \frac{x^2}{2}\right)^2 =$$

$$= \left(x + \frac{x^2}{2} + o(x^2)\right)^2 =$$

$$= \left(x + \frac{x^2}{2} + o(x^2)\right)\left(x + \frac{x^2}{2} + o(x^2)\right)$$

$$o(x^3)$$

$$= o(x^3) + \left(x + \frac{x^2}{2}\right)^2 =$$

$$= o(x^3) + x^2 + x^3$$

$$e^{(x+1)\ln(1+x)} = 1 + (x+1)\ln(1+x) + \frac{\left((x+1)\ln(1+x)\right)^2}{2} +$$

$$+ \frac{\left((x+1)\ln(1+x)\right)^3}{6} + o(x^3)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + o(x^3) +$$

$$+ \frac{1}{2}\left(x^2 + x^3 + o(x^3)\right) + \frac{1}{6}\left(x^3 + o(x^3)\right) + o(x^3)$$

$$= 1 + x + x^2 + \frac{1}{2}x^3 + o(x^3)$$

$$\frac{(x+1)^{x+1} - e^{x^2} - x}{\sin x^3} =$$

$$= \frac{\cancel{1} + \cancel{x} + \cancel{x^2} + \frac{x^3}{2} - \cancel{1} - \cancel{x^2} - \cancel{x} + o(x^3)}{x^3 + o(x^3)} =$$

$$= \frac{\frac{1}{2} + \frac{o(x^3)}{x^3}}{1 + \frac{o(x^3)}{x^3}} \xrightarrow[x \to 0]{} \frac{1}{2}$$