2. Dicembre. 2021

COEFFICIENTE BINOMIALE "GENERALIZZATO":

$\alpha \in \mathbb{R}, \quad K \in \mathbb{N}:$

$$\binom{\alpha}{K} := \frac{\overbrace{\alpha \cdot (\alpha-1) \cdot (\alpha-2) \cdot \ldots \cdot (\alpha-K+1)}^{K \text{ fattori}}}{K!}$$

$$\binom{\alpha}{0} := 1$$

OSS.! Se $\alpha \in \mathbb{N}: \quad K \leq \alpha$

la definizione precedente coincide con l'usuale COEFFICIENTE BINOMIALE

Esempio:

$\alpha = \frac{1}{2}$ $\qquad K = 3$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) \cdot \left(\frac{1}{2} - 2\right)}{3!} =$$

$$= \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{\cancel{3}}{2}\right)}{\cancel{3} \cdot 2 \cdot 1} =$$

$$= \frac{1}{16}$$

$\alpha = \frac{1}{2}$ $\qquad K = 1$

$$\binom{\frac{1}{2}}{1} = \frac{\frac{1}{2}}{1!} = \frac{1}{2}$$

$$\alpha = \frac{1}{2}$$

$$k = 2$$

$$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right)}{2!} = \frac{\frac{1}{2} \cdot \left(-\frac{1}{2}\right)}{2} = -\frac{1}{8}$$

$$\binom{-1}{j} = \frac{-1 \cdot (-1-1) \cdot (-1-2) \cdot \ldots \cdot (-1-j+1)}{j!}$$

$$= \frac{-1 \cdot (-2) \cdot (-3) \cdot \ldots \cdot (-j)}{j!}$$

$$= \frac{(-1)^j \cdot j!}{j!} = (-1)^j$$

$$\binom{\alpha}{1} = \frac{\alpha}{1!} = \alpha$$

$$\boxed{\begin{gathered}\alpha \in \mathbb{R} \, : \quad \alpha \neq 0 \\ (1+t)^{\alpha} = \sum_{j=0}^{n} \binom{\alpha}{j} t^{j} + o(t^{n})\end{gathered}}$$

<u>se $n = 3$</u> :

$$(1+t)^{\alpha} = \binom{\alpha}{0} t^{0} + \binom{\alpha}{1} t^{1} + \binom{\alpha}{2} t^{2} + \binom{\alpha}{3} t^{3} + o(t^{3})$$

$$= 1 + \alpha t + \frac{\alpha(\alpha-1)}{2} t^{2} + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} t^{3} + o(t^{3})$$

Es.:

$$\alpha = \frac{1}{2}$$

$$\sqrt{1+t} = (1+t)^{\frac{1}{2}} =$$

$$= \sum_{j=0}^{n} \binom{\frac{1}{2}}{j} t^j + o(t^n)$$

$n = 3$:

$$\sqrt{1+t} = \sum_{j=0}^{3} \binom{\frac{1}{2}}{j} t^j + o(t^3)$$

$$\sqrt{1+t} = \sum_{j=0}^{3} \binom{\frac{1}{2}}{j} t^j + o(t^3)$$

$$= \binom{\frac{1}{2}}{0} t^0 + \binom{\frac{1}{2}}{1} t +$$

$$+ \binom{\frac{1}{2}}{2} t^2 + \binom{\frac{1}{2}}{3} t^3 + o(t^3)$$

$$= 1 + \frac{1}{2} t + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} t^2 +$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} t^3 + o(t^3)$$

$$= 1 + \frac{1}{2} t - \frac{1}{8} t^2 + \frac{1}{16} t^3 + o(t^3)$$

$$\frac{1}{1+t} = (1+t)^{-1} =$$

$$= \sum_{j=0}^{n} \binom{-1}{j} t^j + t^n$$

Es.:

Abbiamo visto prima: $\binom{-1}{j} = (-1)^j$

$n = 4$:

$$\frac{1}{1+t} = \sum_{j=0}^{4} \binom{-1}{j} t^j + o(t^4)$$

$$= \binom{-1}{0} t^0 + \binom{-1}{1} t + \binom{-1}{2} t^2 + \binom{-1}{3} t^3 + \binom{-1}{4} t^4 + o(t^4)$$

$$= 1 - t + t^2 - t^3 + t^4 + o(t^4)$$

TAVOLA DEGLI SVILUPPI DI TAYLOR,
DI PUNTO INIZIALE $x_0 = 0$,
DI ALCUNE FUNZIONI ELEMENTARI

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + o(x^8)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + o(x^n)$$

$$\cancel{\arcsin x = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots + \frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1} + o(x^{2n+2})}$$

$$\cancel{\arccos x = \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3}{40}x^5 - \ldots - \frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1} + o(x^{2n+2})}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\text{con } \binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$$

Per esempio :

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots + (-1)^{n+1}\frac{(2n-3)!!}{(2n)!!}x^n + o(x^n) \quad (n \geq 1)$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{8}x^2 - \frac{15}{48}x^3 + \ldots + (-1)^n \frac{(2n-1)!!}{(2n)!!}x^n + o(x^n)$$

ALCUNI ESEMPI:

(1) Sviluppo al III ordine:

- $\sin x = x - \frac{x^3}{6} + o(x^3)$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$
- $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$

(2) Sviluppo al II ordine:

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

ALCUNI CASI PARTICOLARI

SIN(x), COS(x)

$$\sin x = x + o(x)$$

$$= x + 0 \cdot x^2 + o(x^2) = x + o(x^2)$$

$$\begin{cases} = x - \dfrac{x^3}{6} + o(x^3) \\ = x - \dfrac{x^3}{6} + 0 \cdot x^4 + o(x^4) = \end{cases}$$

$$= x - \frac{x^3}{6} + o(x^4)$$

$$\cos x = 1 + o(1)$$

$$= 1 + 0 \cdot x + o(x) = 1 + o(x)$$

$$\begin{cases} = 1 - \dfrac{x^2}{2} + o(x^2) \\ = 1 - \dfrac{x^2}{2} + 0 \cdot x^3 + o(x^3) \end{cases}$$

$$= 1 - \frac{x^2}{2} + o(x^3)$$

Oss.:

$$e^x = 1 + x + o(x)$$

$$\cancel{e^x = 1 + x + o(x^2)}$$ NO!!

Come capire l'ordine di prontezza di un infinitesimo?

1. $e^x - 1 - x = \,?$

Se sviluppo e^x al I ordine:

$$e^x = 1 + x + o(x)$$

$$\downarrow$$

$$e^x - 1 - x = (1 + x + o(x)) - 1 - x =$$

$$= o(x)$$ NO

Lo sviluppo al II ordine:

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$$

$\Rightarrow$

$$e^x - 1 - x =$$

$$= \left(1 + x + \frac{x^2}{2} + o(x^2)\right) - 1 - x =$$

$$= \frac{x^2}{2} + o(x^2)$$

OK

Al III ordine:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)$$

$$e^x - 1 - x = \left(\cancel{1} + \cancel{x} + \frac{x^2}{2} + \frac{x^3}{6} + o(x^3)\right) \cancel{-1} \cancel{-x}$$

$$= \frac{x^2}{2} + \cancel{\frac{x^3}{6}} + o(x^3)$$

(2) $e^x + e^{-x} - 2 = ?$

Se sviluppiamo e^x, e^{-x} al I ordine: $\left(e^t = 1 + t + o(t) \right)$

$$\left(e^{t} = 1 + t + o(t) \right)$$

$$e^{x} = 1 + x + o(x) \qquad (t = x)$$

$$e^{-x} = 1 + (-x) + o(-x) \qquad (t = -x)$$

$$o(x)$$

$$= 1 - x + o(x)$$

$$e^{x} + e^{-x} - 2 =$$

$$= \left(\cancel{1} + x + o(x)\right) + \left(\cancel{1} - x + o(x)\right) - \cancel{2}$$

$$= o(x)$$

<u>**NO**</u>

$$\left(e^t = 1 + t + \frac{t^2}{2} + o(t^2) \right)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \qquad (t = x)$$

$$e^{-x} = 1 + (-x) + \frac{(-x)^2}{2} + o((-x)^2) \qquad (t = -x)$$

$$= 1 - x + \frac{x^2}{2} + o(x^2)$$

$$e^x + e^{-x} - 2 =$$

$$= \left(\cancel{1} + x + \frac{x^2}{2} + o(x^2)\right) + \left(\cancel{1} - x + \frac{x^2}{2} + o(x^2)\right)$$

$$- 2$$

$$= x^2 + o(x^2) \qquad \checkmark$$

Esercizio (sui limiti)

Metodo di calcolo del limite:

$$\lim_{x \to 0} \frac{f(x)}{g(x)}$$

- si analizzano numeratore e denominatore, e si inizia a sviluppare il più semplice fra i due;
- se il più semplice è il denominatore, si deve stabilire per quale $m \in \mathbb{N}$:

$$g(x) = \alpha x^m + o(x^m) \qquad (\alpha \neq 0)$$

- si sviluppa f almeno all'ordine m.

EJEMPIO (per la funzione $o(x)$)

- $e^x - 1 = o(x)$

$$e^x = 1 + x + o(x)$$

$$e^x - 1 = \boxed{x} + o(x)$$

$$\downarrow$$

$$m = 1$$

- $(\cos^3 x - 1)^2 = o(x)$

$$\cos x = 1 - \frac{x^2}{2} + o(x^2)$$

$$(\cos x)^3 = \left(1 - \frac{x^2}{2} + o(x^2)\right)^3 =$$

$$(\cos x)^3 = \left(1 - \frac{x^2}{2} + o(x^2)\right)^3 =$$

$$= \left(1 - \frac{x^2}{2} + o(x^2)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right) =$$

$$\Longrightarrow o(x^2)$$

$$= \left(1 - \frac{x^2}{2} + o(x^2)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right) =$$

$$= \left(\overset{A}{1} \overset{B}{- \frac{x^2}{2}}\right)^3 + o(x^2)$$

$$= A^3 + 3A^2B + 3AB^2 + B^3 + o(x^2)$$

$$= 1^3 + 3\left(-\frac{x^2}{2}\right) + \underbrace{3\cdot\left(-\frac{x^2}{2}\right)^2 + \left(-\frac{x^2}{2}\right)^3}_{o(x^2)} + o(x^2)$$

$$= 1 - \frac{3}{2}x^2 + o(x^2)$$

$$(\cos^3 x - 1)^2 = \left(o(x^2) + \cancel{1} - \frac{3}{2}x^2 \cancel{-1} \right)^2 =$$

$$= \left(\overset{A}{-\frac{3}{2}x^2} + \overset{B}{o(x^2)} \right)^2 = \qquad A^2 + 2AB + B^2$$

$$= \left(-\frac{3}{2}x^2 \right)^2 + 2\left(-\frac{3}{2}x^2 \right) o(x^2) + \left(o(x^2) \right)^2$$

$$= \frac{9}{4}x^4 + o(x^4)$$

$m = 4$

$$\lim_{x \to 0} \frac{\cos x - 1 + \frac{x^2}{2} + x^4}{x^4}$$

$\Rightarrow n = 4$

Si deve sviluppare il numeratore almeno al $\overline{\underline{IV}}$ ordine:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + o(x^4)$$

$$\frac{\cos x - 1 + \frac{x^2}{2} + x^4}{x^4} =$$

$$= \frac{\cancel{1} - \cancel{\frac{x^2}{2}} + \frac{x^4}{24} + o(x^4) - \cancel{1} + \cancel{\frac{x^2}{2}} + x^4}{x^4} =$$

$$= \frac{\frac{25}{24} x^4 + o(x^4)}{x^4} =$$

$$= \frac{25}{24} + \frac{o(x^4)}{x^4} \xrightarrow[x \to 0]{} \frac{25}{24}$$

$$\lim_{x \to 0} \frac{e^{\sin x} - 1 - x - \frac{x^2}{2}}{x^3}$$

sviluppiamo il numeratore al III ordine (cioè, $o(x^3)$)

$$e^{\sin x} = t$$

$$\left(e^t = \sum_{j=0}^{n} \frac{t^j}{j!} + o(t^n) \right)$$

$$x \longrightarrow 0 \implies t = \sin x \longrightarrow 0$$

$$\sin x = x + o(x) \implies \sin x \simeq x$$

$$\implies t = \sin x \simeq x$$

$$o(t^n) = o(\sin^n x) = o(x^n) \implies n = 3$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3)$$

$$e^{\sin x} = 1 + \sin x + \frac{1}{2} \sin^2 x + \frac{1}{6} \sin^3 x + o(x^3)$$

$$e^{\sin x} = 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\sin^2 x = \left(x + o(x^2)\right)^2 =$$

$$= x^2 + 2x \cdot o(x^2) + \left(o(x^2)\right)^2$$

$$= x^2 + o(x^3) + o(x^4) = x^2 + o(x^3)$$

$$\sin^3 x = \left(x + o(x^2)\right)^3 =$$

$$= x^3 + 3x^2 o(x^2) + \ldots$$

$$= x^3 + o(x^4) = x^3 + \underline{o(x^3)}$$

$$\sin^2 x = \left(\underline{x + o(x)}\right)^2 =$$

$$= x^2 + \underbrace{2x \cdot o(x)} + \underbrace{(o(x))^2} =$$

$$= \underline{x^2 + \boxed{o(x^2)}} \qquad \underline{\underline{NO}}$$

$o(x^2)$ è un $o(x^3)$? $\underline{\underline{NO}}$

$$\sin^2 x = \left(\underline{x + \cancel{0 \cdot x^2} + o(x^2)}\right)^2 =$$

$$= x^2 + \underbrace{2x\, o(x^2)}_{o} + \underbrace{(o(x^2))^2}_{o} =$$

$$= x^2 + o(x^3) + \cancel{o(x^4)}$$

$$= x^2 + o(x^3) \leftarrow \text{SÌ}$$

$$e^{\sin x} = 1 + \sin x + \frac{1}{2}\sin^2 x + \frac{1}{6}\sin^3 x + o(x^3)$$

$$= 1 + \left(x - \frac{x^3}{6} + o(x^3)\right) + \frac{1}{2}\left(x^2 + o(x^3)\right) + \frac{1}{6}\left(x^3 + o(x^3)\right) + o(x^3)$$

$$= 1 + x + \frac{x^2}{2} + o(x^3)$$

$$\frac{e^{\sin x} - 1 - x - \frac{x^2}{2}}{x^3} =$$

$$= \frac{\cancel{1} + \cancel{x} + \cancel{\frac{x^2}{2}} + o(x^3) - \cancel{1} - \cancel{x} - \cancel{\frac{x^2}{2}}}{x^3} =$$

$$= \frac{o(x^3)}{x^3} \xrightarrow[x \to 0]{} 0$$

$$f(n)^{\phi(n)} = e^{\boxed{\phi(n) \ln f(n)}}$$

$$\lim_{x \to 0} \frac{\sin^2 x - x^2}{e^{x^4} - 1} = ?$$

$t = x^4$

$$\left(e^t = 1 + t + o(t) \right)$$

Più semplice $\longrightarrow$ il denominatore

$$e^{x^4} = 1 + x^4 + o(x^4)$$

$$\Longrightarrow \quad e^{x^4} - 1 = x^4 + o(x^4)$$

Si deve sviluppare il numeratore al $\overline{\text{IV}}$ ordine:

$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3) \right)^2 =$$

$$= \left(x - \frac{x^3}{6} + o(x^3) \right) \left(x - \frac{x^3}{6} + o(x^3) \right) =$$

$x \cdot o(x^3) = o(x^4)$

$-\frac{x^3}{6} o(x^3) = o(x^6)$

$(o(x^3))^2 = o(x^6)$

$$\sin^2 x = \left(\underline{x + o(x^2)}\right)^2 =$$

$$= x^2 + \underline{2x \cdot o(x^2)} + \left(o(x^2)\right)^2 =$$

$$= x^2 + \underline{o(x^3)} + \cancel{o(x^4)}$$

$$= \underline{x^2 + o(x^3)}$$

No

$$\sin^2 x = \Big(\overbrace{x - \frac{x^3}{6}}^{A} + \overbrace{o(x^3)}^{B}\Big)^2$$

$$= \left(x - \frac{x^3}{6}\right)^2 + \overbrace{2\left(x - \frac{x^3}{6}\right)\cdot o(x^3)}^{= o(x^4)}$$

$$+ \underbrace{\big(o(x^3)\big)^2}_{o(x^4)}$$

$$= x^2 + 2x\left(-\frac{x^3}{6}\right) + o(x^4)$$

$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$\sin^2 x = \left(x - \frac{x^3}{6}\right)^2 + o(x^4)$$

$$= x^2 + 2x \cdot \left(-\frac{x^3}{6}\right) + \cancel{\left(-\frac{x^3}{6}\right)^2} + o(x^4) =$$

$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$\frac{\sin^2 x - x^2}{e^{x^4} - 1} = \frac{\cancel{x^2} - \frac{x^4}{3} + o(x^4) - \cancel{x^2}}{\cancel{1} + x^4 + o(x^4) - \cancel{1}} =$$

$$= \frac{-\frac{1}{3} + \frac{o(x^4)}{x^4}}{1 + \frac{o(x^4)}{x^4}} \xrightarrow[x \to 0]{} -\frac{1}{3}$$

$$\lim_{x \to 0} \frac{\sin^2 x - x^2}{e^{x^4} - 1} = -\frac{1}{3}$$