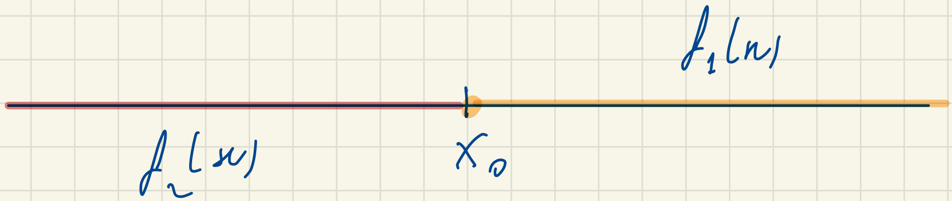
18. Novembre. 2021

$$g(x) = \begin{cases} f_1(x) & x \geq x_0 \\ f_2(x) & x < x_0 \end{cases}$$



1 $x < x_0$:

$g(x)$ è derivabile in $x < x_0$ $\iff$

$\iff$ $f_2(x)$ è derivabile in $x < x_0$

$x > x_0$:

$g(x)$ è derivabile in $x > x_0$ $\iff$

$\iff$ $f_1(x)$ è derivabile in x_0

② $x = x_0$!

Si calcolano, se esistono:

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f_1(x) - f_1(x_0)}{x - x_0}$$

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f_2(x) - f_2(x_0)}{x - x_0}$$

(se non esiste uno dei suddetti limiti $\Rightarrow f$ non è derivabile in x_0)

Se esistono entrambi, ma almeno è $+\infty$ o $-\infty$ $\Rightarrow f$ non è derivabile in x_0

Esistono finiti entrambi i limiti sopra:

f è derivabile in x_0

$$\Longleftrightarrow \quad f'_+(x_0) = f'_-(x_0)$$

In tal caso:

$$f'(x_0) = f'_+(x_0) = f'_-(x_0)$$

OSSERVAZIONE:

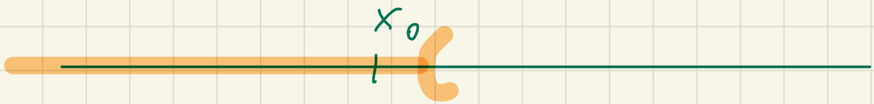
$$f(x) = \begin{cases} f_1(x) & x \geq x_0 \\ f_2(x) & x < x_0 \end{cases}$$

Se f_1 è una funzione derivabile in una regione:

(dove x_0 è un punto interno)

$$\Longrightarrow \quad f'_+(x_0) = f'_1(x_0)$$

Se f_2 è una funzione derivabile in una regione:



(dove x_0 è un punto interno)

$$\Longrightarrow \quad f'_-(x_0) = f'_2(x_0)$$

Esercizi:

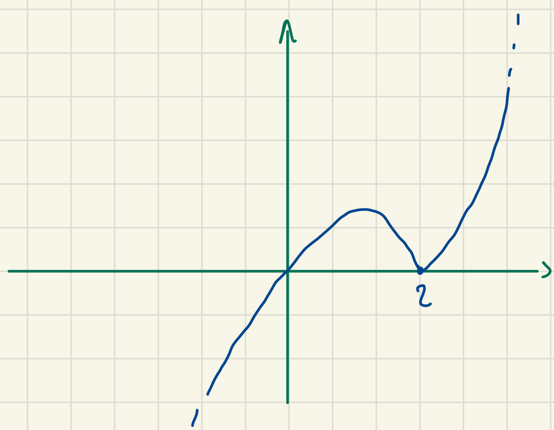
Disegnare il grafico di:

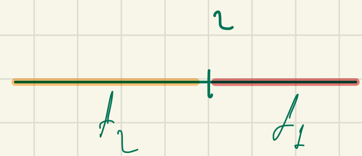
(1) $f(x) = x\,|x-2|$

(suggerim.: eliminare il valore assoluto

$$|x-2| = \begin{cases} x-2 & \text{se } x \geq 2 \\ -(x-2) & \text{se } x < 2 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} \overbrace{x^2-2x}^{=f_1(x)} & \text{se } x \geq 2 \\ \underbrace{-x^2+2x}_{=f_2(x)} & \text{se } x < 2 \end{cases}$$)

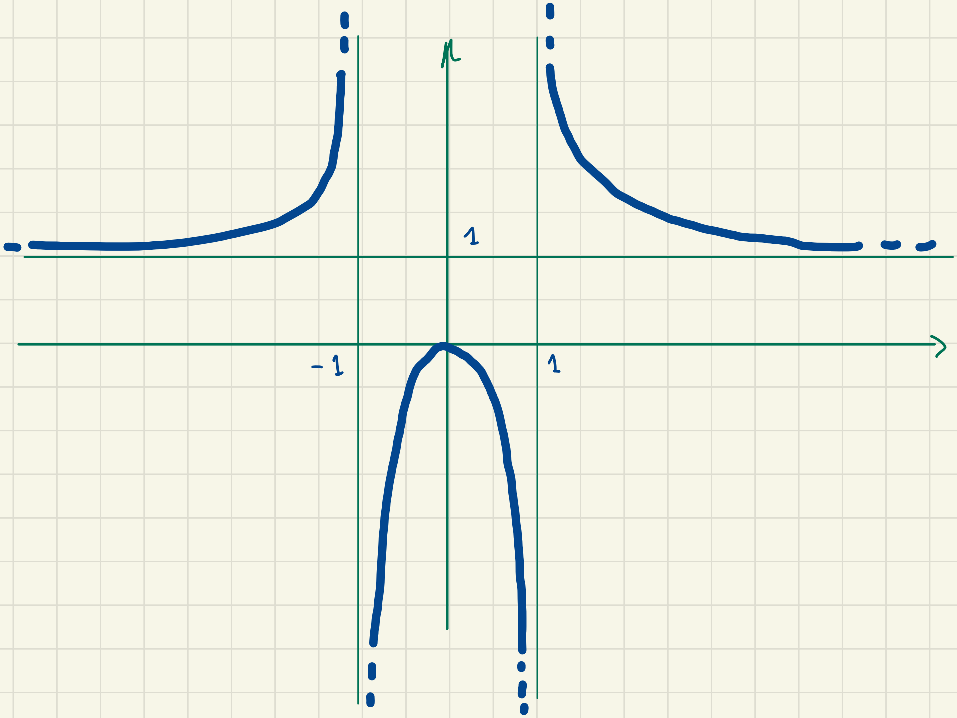




Il punto $x = 2$ è di NON DERIVABIL. di f :

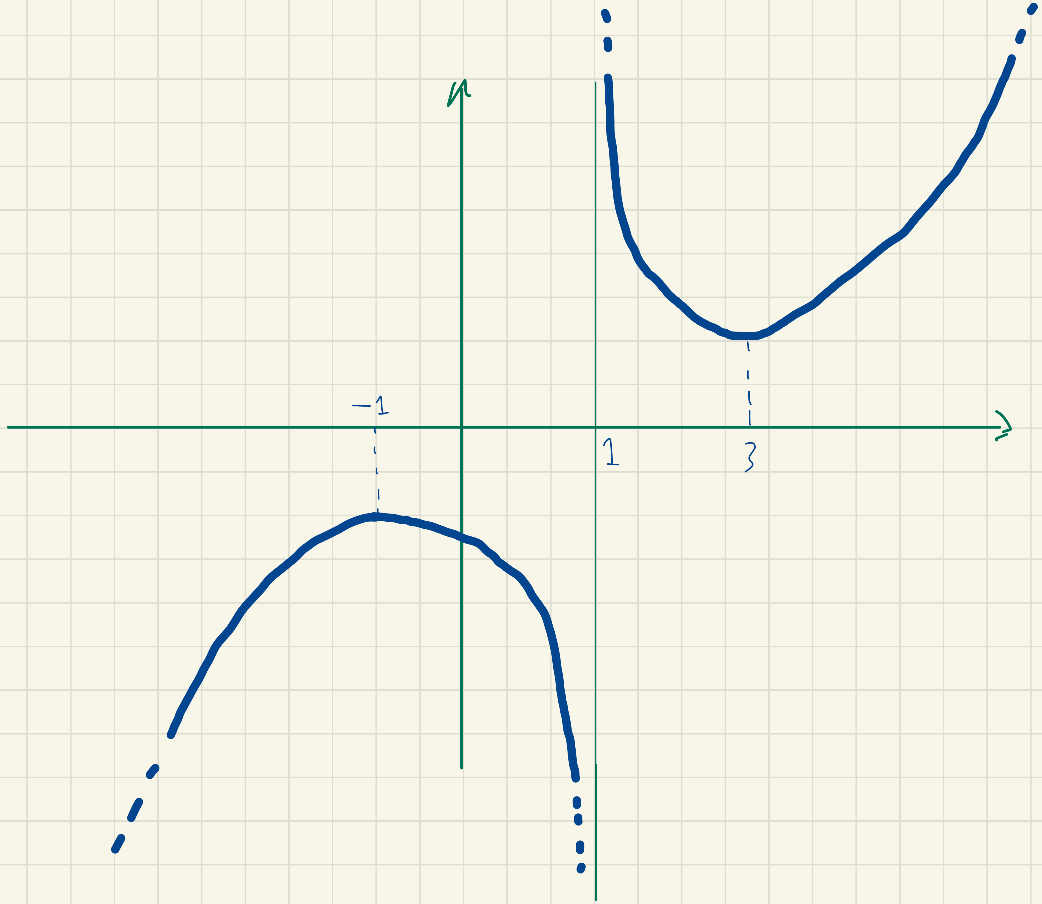
$$f'_{-}(2) = f_2'(2) = -2$$
$$\neq$$
$$f'_{+}(2) = f_1'(2) = 2$$

(2) $f(x) = \dfrac{x^2}{x^2 - 1}$

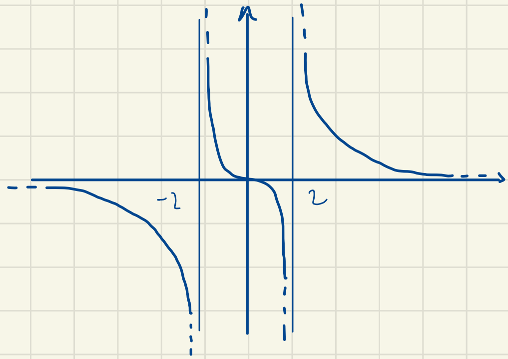


③

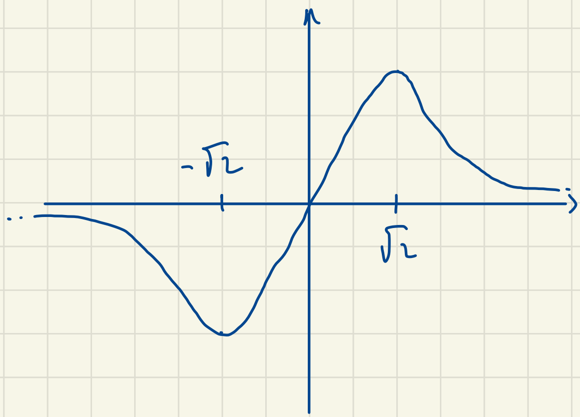
$$f(x) = \frac{x^2 + 3}{x - 1}$$



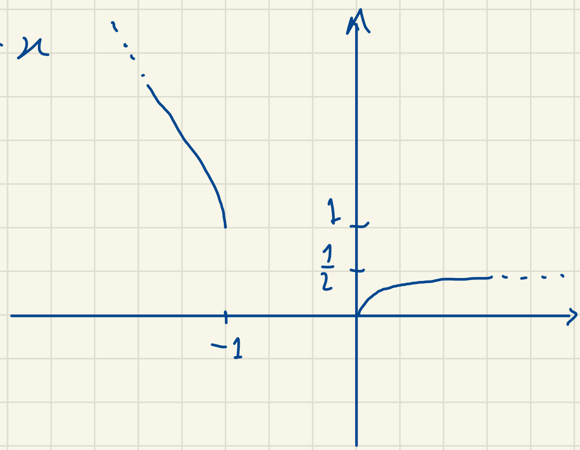
4 $\quad f(x) = \dfrac{x}{x^2 - 4}$



5 $\quad f(x) = \dfrac{x}{x^2 + 2}$



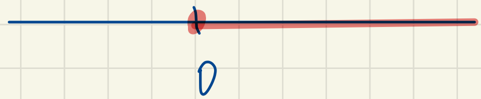
6 $\quad f(x) = \sqrt{x^2 + x} - x$



$$f(x) = \sqrt{x}$$

$$D(f) = \{ x \in \mathbb{R} \mid x \geq 0 \}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$



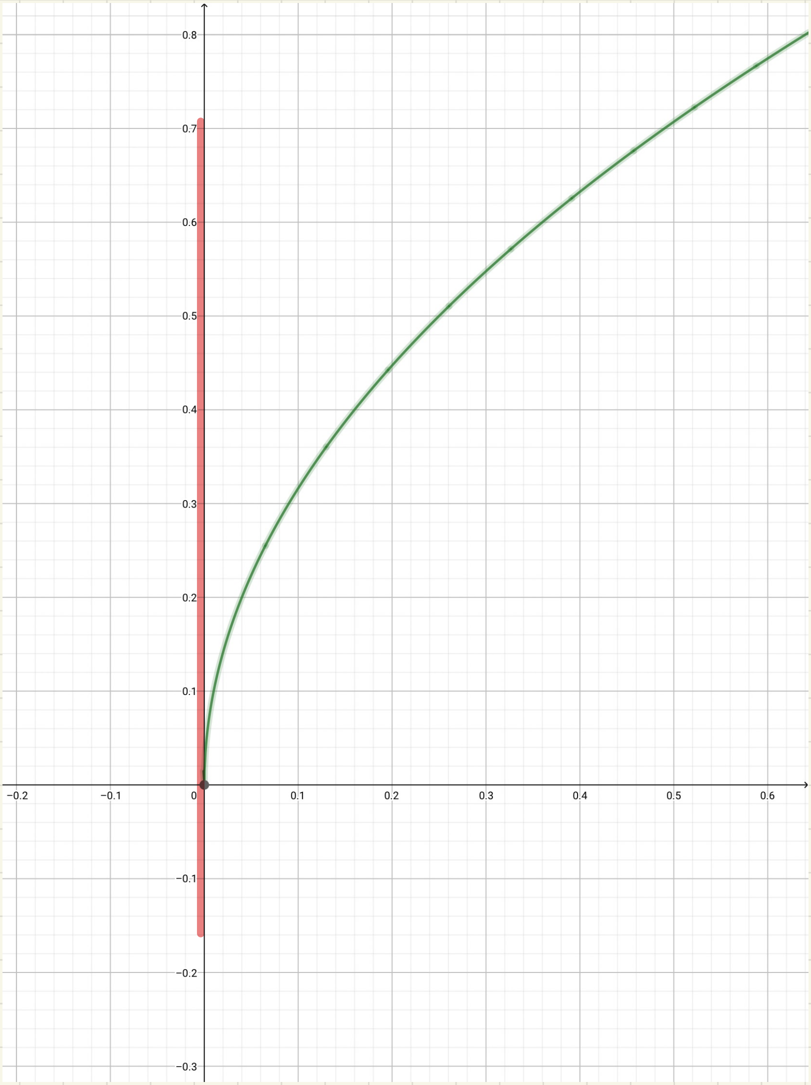
f è derivabile $\iff x > 0$

($\exists\, f'_+(0)$?

$$f_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} =$$

$$= \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} =$$

$$= +\infty$$)



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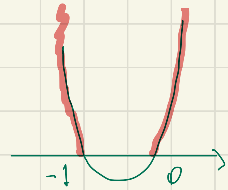
(6)

$$f(x) = \sqrt{x^2+x} - x$$

$$D(f) = \{ x \in \mathbb{R} \mid \underbrace{x^2 + x}_{x(x+1)} \geq 0 \}$$



$$= \{ x \in \mathbb{R} \mid x \leq -1 \ \lor \ x \geq 0 \}$$

$$\lim_{x \to +\infty} \sqrt{x^2+x} - x$$

$$\left(\sqrt{A} - B = \frac{(\sqrt{A}-B)(\sqrt{A}+B)}{\sqrt{A}+B} \right)$$

$$= \lim_{x \to +\infty} \frac{\cancel{x^2} + x - \cancel{x^2}}{\sqrt{x^2+x} + x} = \qquad \frac{A - B^2}{\sqrt{A}+B}$$

$$= \lim_{x \to +\infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1} = \frac{1}{2}$$

$$\lim_{x \to -\infty} \sqrt{x^2+x} - x = +\infty$$

$$\lim_{x \to -1^-} \sqrt{x^2+x} - x = f(1) = 1$$

$$\lim_{x \to 0^+} \sqrt{x^2+x} - x = f(0) = 0$$

$$f'(x) = \frac{1}{2\sqrt{x^2+x}} \cdot (2x+1) - 1 =$$

$$= \frac{2x+1}{2\sqrt{x^2+x}} - 1 =$$

$$= \frac{2x+1 - 2\sqrt{x^2+x}}{2\sqrt{x^2+x}}$$

$$2x+1 - 2\sqrt{x^2+x} = 0$$

$$2x+1-2\sqrt{x^2+x}=0$$

$$\begin{cases} 2\sqrt{x^2+x} = 2x+1 \\ x^2+x>0 \\ 2x+1 \geq 0 \end{cases}$$

$$4(x^2+x) = 4x^2+4x+1$$

$$\cancel{4x^2}+\cancel{4x} = \cancel{4x^2}+\cancel{4x}+1$$

$$0=1 \qquad \text{Imp.}$$

$$f' \geq 0 \iff 2x+1-2\sqrt{x^2+x} > 0$$

$$\begin{cases} 2\sqrt{x^2+x} < 2x+1 \\ x^2+x>0 \\ 2x+1 \geq 0 \end{cases}$$

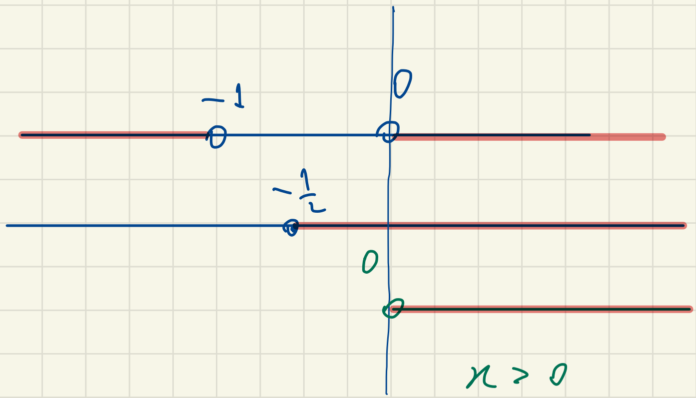
$$\begin{cases} 2\sqrt{x^2+x} < 2x+1 \\ x^2+x > 0 \\ 2x+1 \geq 0 \end{cases}$$

$$\cancel{4x^2} + \cancel{4x} < \cancel{4x^2} + \cancel{4x} + 1$$

$$0 < 1 \qquad \text{Vera}$$

$$\begin{cases} x^2+x > 0 \\ 2x+1 \geq 0 \end{cases}$$

$$x \geq -\frac{1}{2}$$



$$x > 0$$

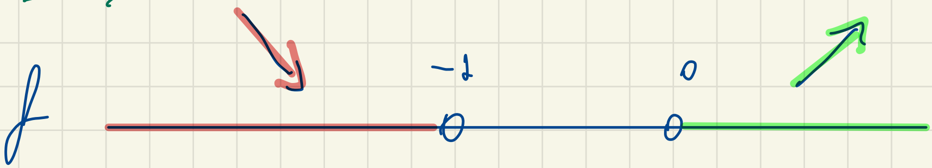
$f' > 0$ per $x > 0$

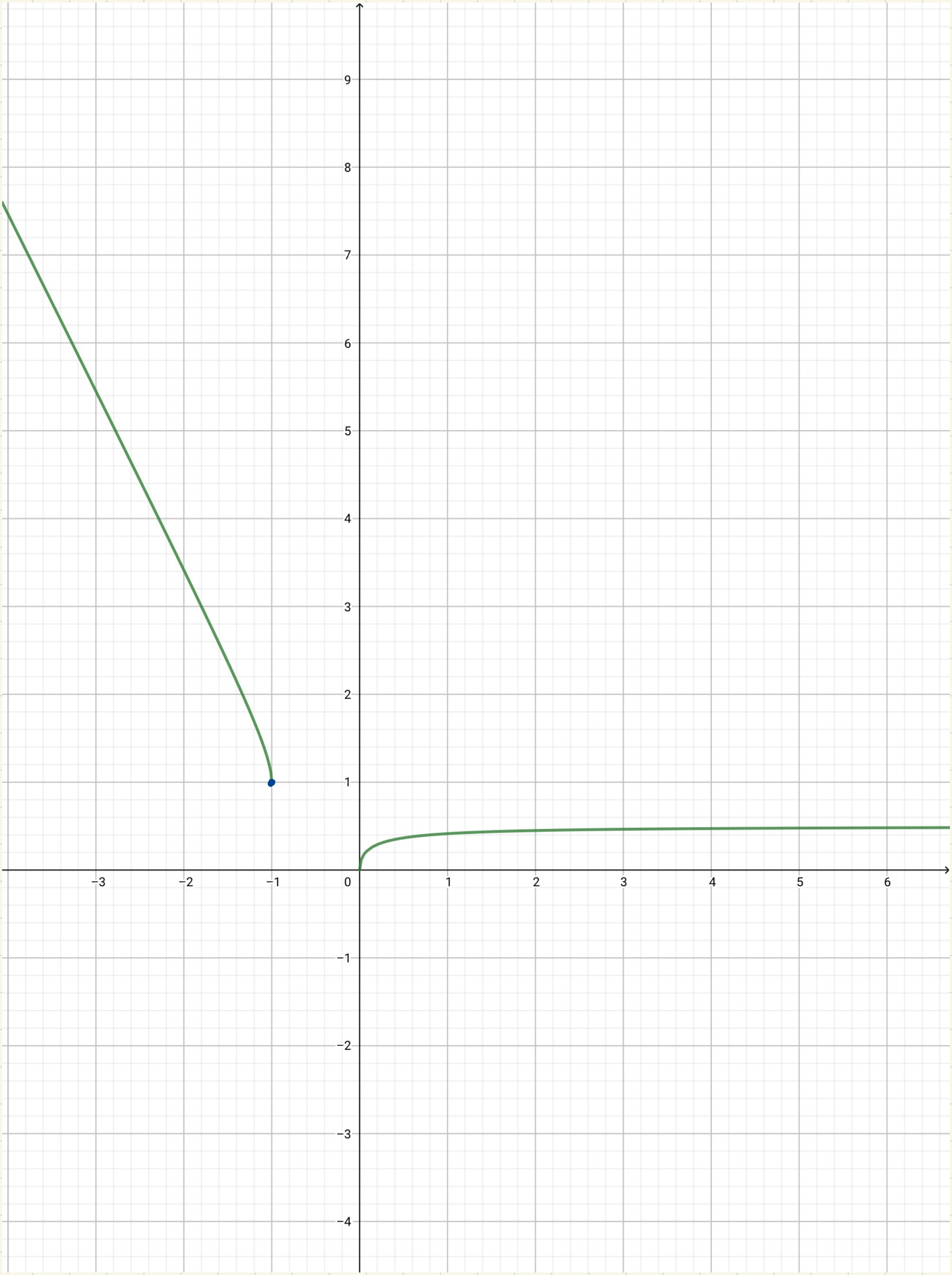
$f' < 0$ per $x < -1$

$f' > 0$ se $x > 0$

$f' < 0$ se $x < -1$

$\Longrightarrow$





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Esercizio: 7

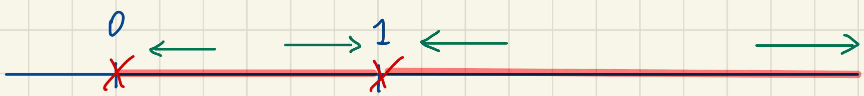
$$f(x) = x \, e^{\frac{1}{\ln x}}$$

Determinare $\operatorname{Im} f$ e discutere il $\# S$ di $f(x) = \lambda$.

1

$$D(f) = \{ x \in \mathbb{R} \mid x > 0 ,\ \ln x \neq 0 \}$$

$$= \{ x \in \mathbb{R} \mid x > 0 ,\ x \neq 1 \}$$



$\longrightarrow$ f non è PARI e non è DISPARI

③

$$\lim_{x\to 0^+} x\, e^{\boxed{\frac{1}{\ln x}}} = 0\cdot e^{0} = 0$$

$x \to 0^+$, $\ln x \to -\infty$, $\frac{1}{\ln x} \to 0$

$$\lim_{x\to 1^-} x\, e^{\boxed{\frac{1}{\ln x}}} = 0$$

$x \to 1$, $\ln x \to 0^-$, $\frac{1}{\ln x} \to -\infty$, $e^{\frac{1}{\ln x}} \to 0$

$$\lim_{x\to 1^+} x\cdot e^{\boxed{\frac{1}{\ln x}}} = +\infty$$

$x \to 1$, $\ln x \to 0^+$, $\frac{1}{\ln x} \to +\infty$

$$\lim_{x\to +\infty} x\cdot e^{\boxed{\frac{1}{\ln x}}} = +\infty$$

$x \to +\infty$, $\ln x \to +\infty$, $\frac{1}{\ln x} \to 0^+$

④ $$f'(x) = 1 \cdot e^{\frac{1}{\ln x}} + \cancel{x} \cdot e^{\frac{1}{\ln x}} \cdot \left(-\frac{1}{(\ln x)^2}\right) \cdot \frac{1}{\cancel{x}}$$

$$= e^{\frac{1}{\ln x}} \cdot \left(1 - \frac{1}{(\ln x)^2}\right) =$$

$$= \underset{\overset{\vee}{0}}{\boxed{\frac{e^{\frac{1}{\ln x}}}{(\ln x)^2}}} \cdot \left(\ln^2 x - 1\right)$$

Si tratta di studiare il segno di f' che coincide con il segno di $\ln^2 x - 1$

$$f'(x) > 0 \iff \underbrace{\ln^2 x}_{t} - 1 > 0$$

$$t^2 - 1 > 0$$

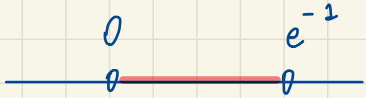
$$\Updownarrow$$

$$t < -1 \lor t > 1$$

$$\ln x < -1$$

$$\downarrow$$

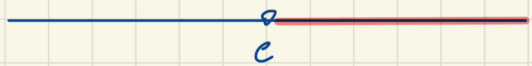
$$\begin{cases} x < e^{-1} \\ x > 0 \end{cases}$$



$$\downarrow$$

$$\ln x > 1$$

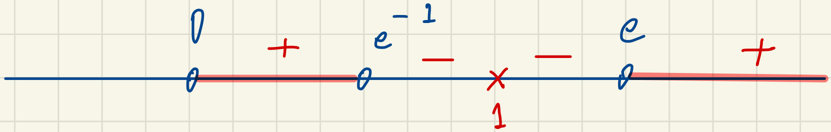
$$x > e$$

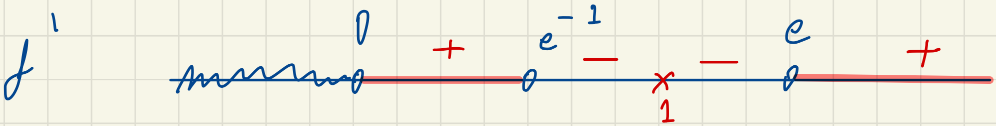


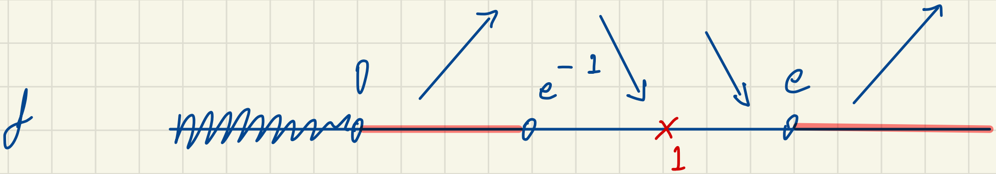
$$f'(e^{-1}) = 0$$

$$f'(e) = 0$$

f'





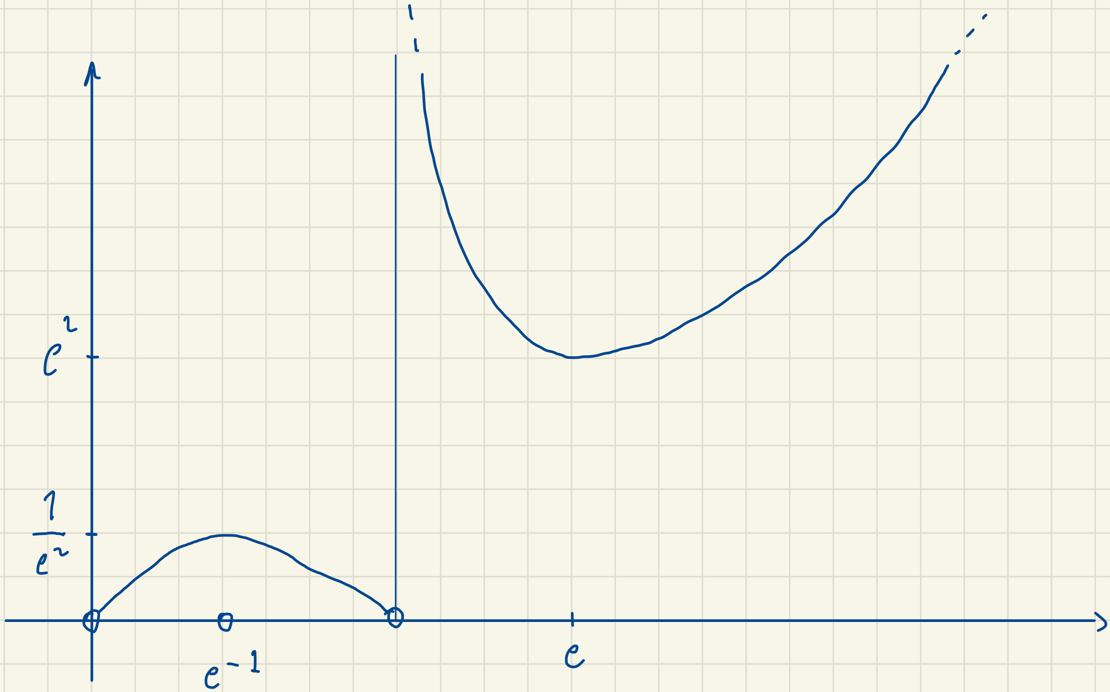


$x = e^{-1}$ p. di max. relativo

$$f(e^{-1}) = e^{-1} \cdot e^{\frac{1}{\ln(e^{-1})}} = \frac{1}{e} \cdot e^{-1} = \frac{1}{e^2}$$

$x = e$ p. di min. relativo

$$f(e) = e \cdot e^{\frac{1}{\ln e}} = e^2$$



$$\operatorname{Im} f = \left]0, \frac{1}{e^2}\right] \cup \left[e^2, +\infty\right[$$

$$f(x) = \lambda$$

$\lambda \leq 0$ $\quad \# S = 0$

$0 < \lambda < \frac{1}{e^2}$ $\quad \# S = 2$

$\lambda = \frac{1}{e^2}$ $\quad \# S = 1$

$\frac{1}{e^2} < \lambda < e^2$ $\quad \# S = 0$

$\lambda = e^2$ $\quad\quad$ $\# \Gamma = 1$

$\lambda > e^2$ $\quad\quad$ $\# \Gamma = 2$

In sintesi:

$$\# \Gamma = 0 \iff \lambda \leq 0 \ \vee \ \frac{1}{e^2} < \lambda < e^2$$

$$\# \Gamma = 1 \iff \lambda = \frac{1}{e^2} \ \vee \ \lambda = e^2$$

$$\# \Gamma = 2 \iff 0 < \lambda < \frac{1}{e^2} \ \vee \ \lambda > e^2$$

Esercizio 8:

$$f(x) = \frac{x}{\ln |x|}$$

$$D(f) = \{ x \in \mathbb{R} \mid |x| > 0 , \ln |x| \neq 0 \}$$

$$= \{ x \neq 0 , |x| \neq 1 \}$$

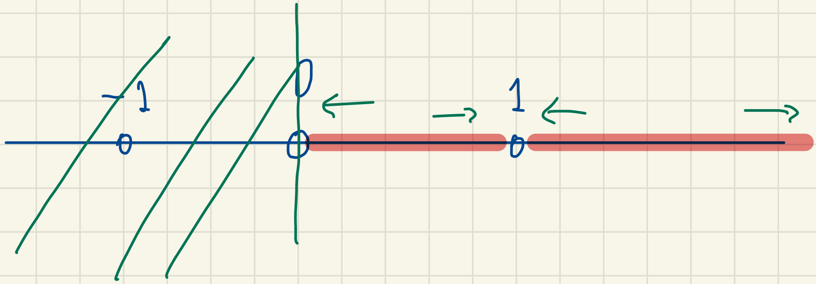
$$= \{ x \neq 0 , x \neq \pm 1 \}$$

$$= \mathbb{R} \setminus \{ 0, -1, +1 \}$$

$$f(-x) = \frac{-x}{\ln |-x|} = - \frac{x}{\ln x} =$$

$$= - f(x)$$

f è una funzione DISPARI



$x > 0$

$\downarrow$

$$f(x) = \frac{x}{\ln x}$$

$$\lim_{x \to 0^+} \frac{\overset{\to 0^+}{x}}{\underset{\to -\infty}{\ln x}} = \lim_{x \to 0^+} \overset{\to 0}{x} \cdot \overset{\to 0}{\frac{1}{\ln x}} = 0$$

$$\lim_{x \to 1^-} \frac{x}{\ln x} = \lim_{x \to 1^-} \overset{\to 1}{x} \cdot \overset{\to -\infty}{\frac{1}{\underset{\to 0^-}{\ln x}}} = -\infty$$

$$\lim_{x \to 1^+} \frac{x}{\ln x} = \lim_{x \to 1^+} \overset{\to 1}{x} \cdot \overset{\to +\infty}{\frac{1}{\underset{\to 0^+}{\ln x}}} = +\infty$$

$$\lim_{x \to +\infty} \frac{x}{\ln x} = +\infty \quad (?)$$

$\downarrow$

vedi più avanti

$$f(x) = \frac{x}{\ln x}$$

$$f'(x) = \frac{\ln x - \frac{1}{x} \cdot x}{(\ln x)^2} =$$

$$= \frac{\ln x - 1}{(\ln x)^2} =$$

$$= \frac{1}{(\ln x)^2} \cdot (\ln x - 1)$$

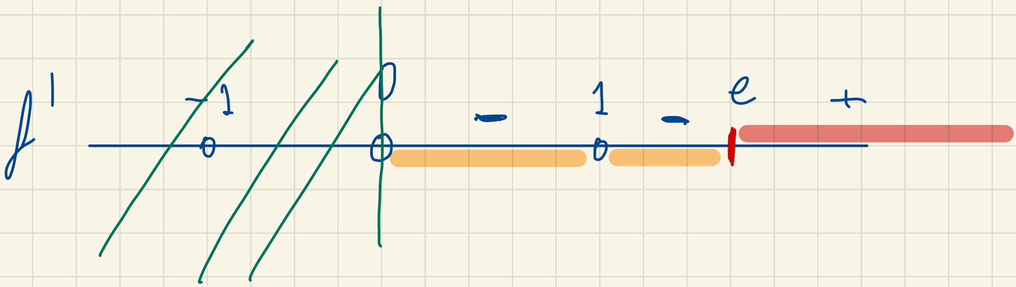
$$f' = 0 \iff \ln x - 1 = 0$$

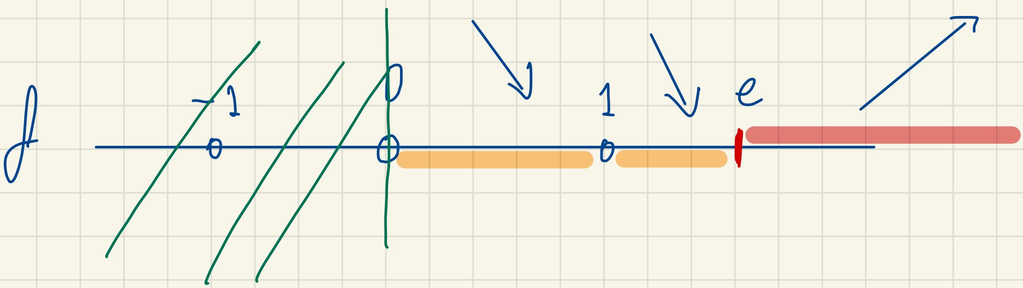
$$x = e$$

$$f' > 0 \iff \ln x - 1 > 0$$

$$\ln x > 1$$

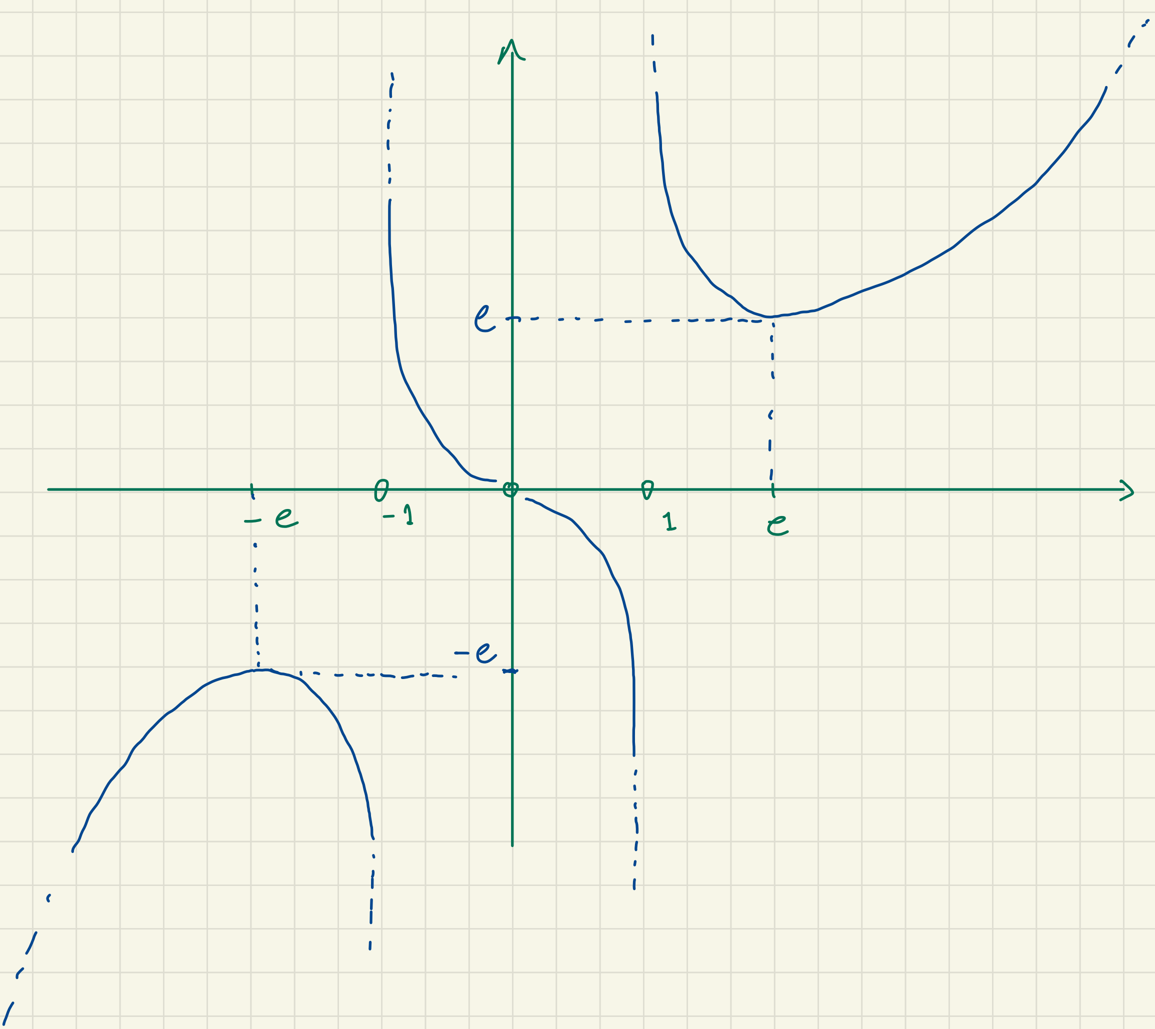
$$x > e$$





$x = e$ p. di min. relativo

$$f(e) = \frac{e}{\ln e} = e$$



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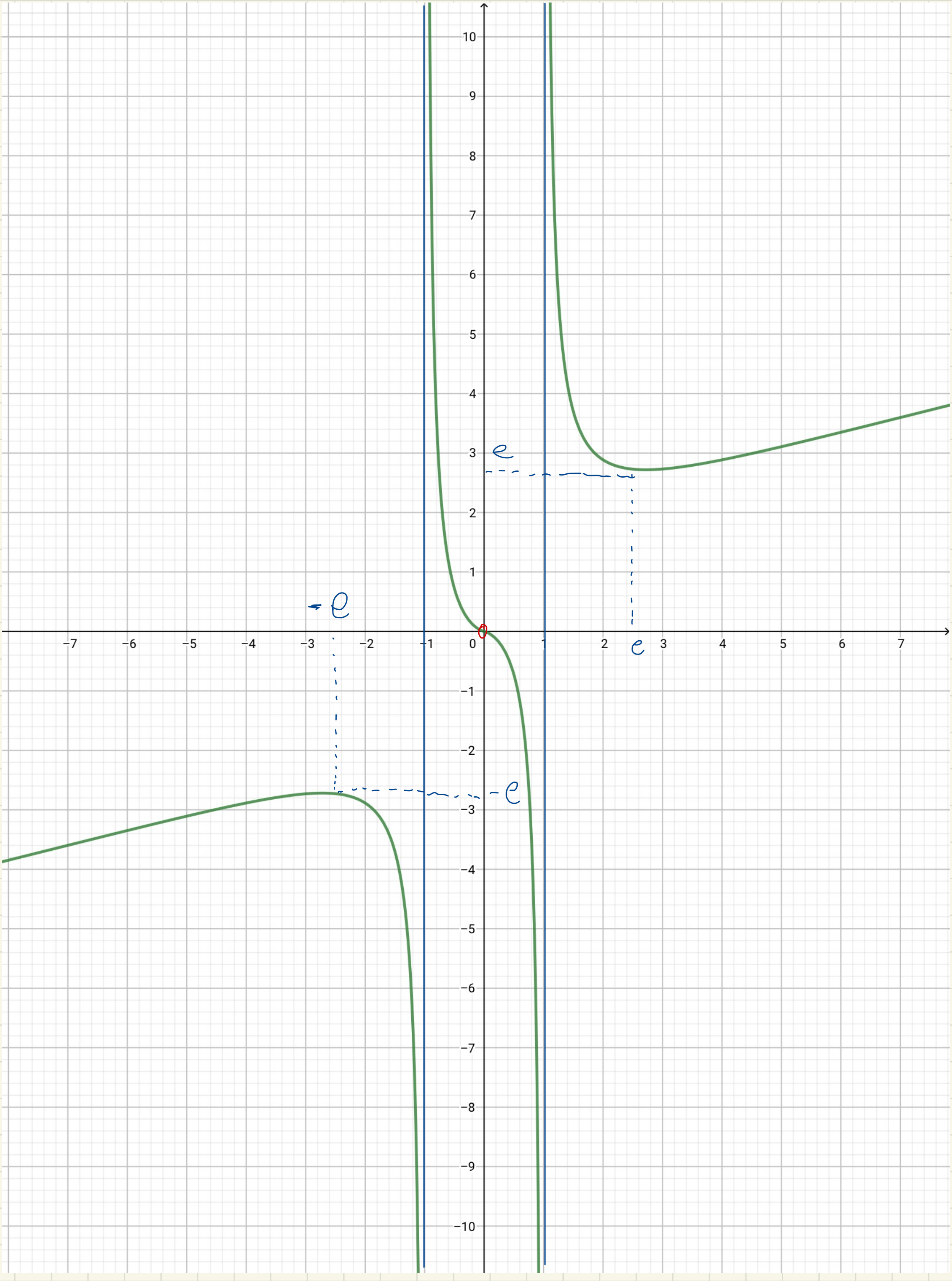

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