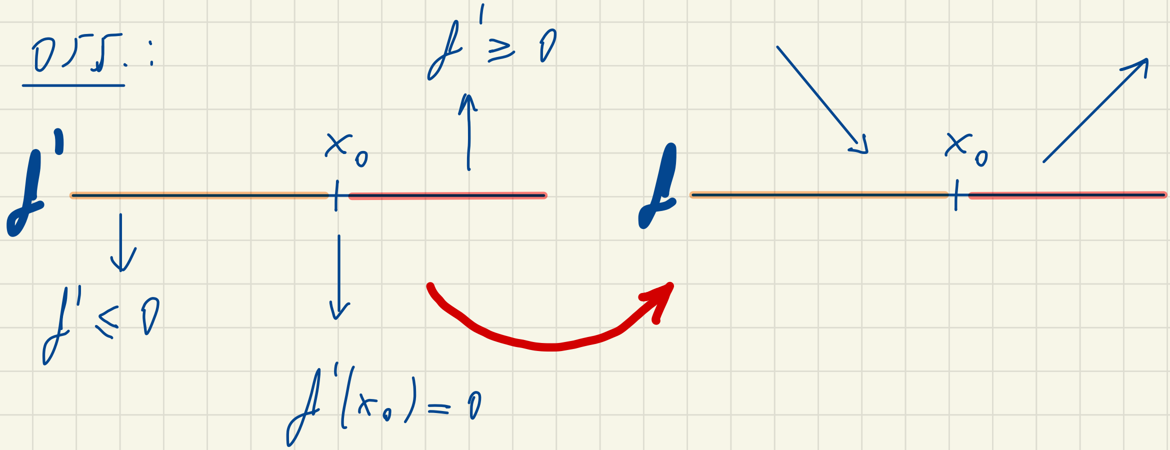
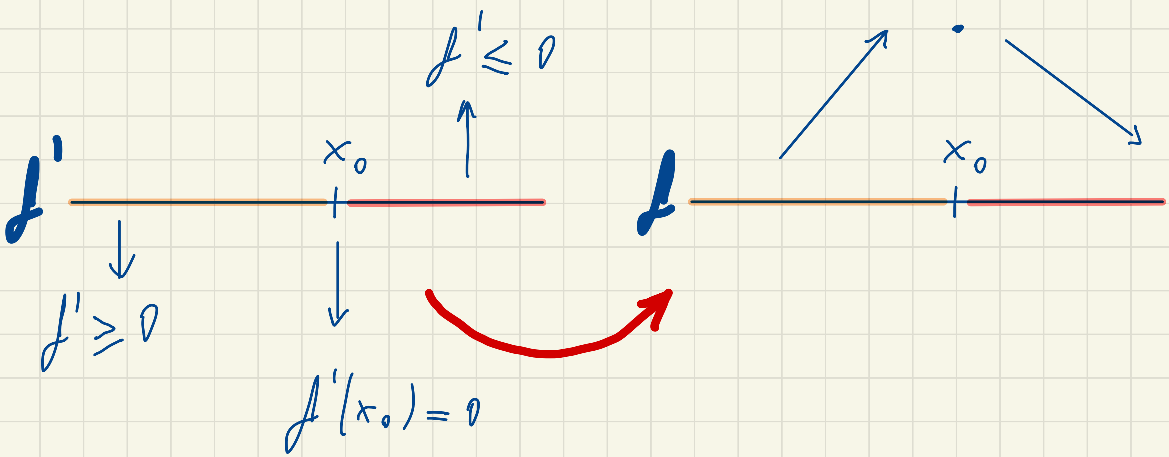
15. Novembre. 2021

Oss.:



$\Longrightarrow$ x_0 punto di minimo relat.



$\Longrightarrow$ x_0 punto di massimo relat.

Oss.:

La condizione precedente è solo sufficiente per garantire un minimo o un massimo relativo.

Esempio:

$$f: \mathbb{R} \longrightarrow \mathbb{R}$$

$$f(x) = \begin{cases} x^2 \cdot \sin^2\left(\frac{1}{x}\right) & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$

$x = 0$ è un p. di minimo (assoluto)

$$f(x) = x^2 \cdot \sin^2\left(\frac{1}{x}\right) \geq 0 = f(0)$$

$\forall x \in \mathbb{R}$

f è derivabile in $x = 0$.

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} =$$

$$= \lim_{x\to 0} \frac{x^2 \cdot \sin^2\left(\frac{1}{x}\right)}{x} =$$

$$= \lim_{x\to 0} x \cdot \sin^2\left(\frac{1}{x}\right) = 0$$

$$\begin{array}{ccccc} 0 & \leq & \left| x \cdot \sin^2\left(\frac{1}{x}\right)\right| & \leq & |x| \\ \Big\downarrow {\scriptstyle x\to 0} & & & & \Big\downarrow {\scriptstyle x\to 0} \\ 0 & & & & 0 \end{array}$$

f è derivabile per $x \neq 0$, poiché $f(x) = x^2 \sin^2\left(\frac{1}{x}\right)$ è composizione di funzioni derivabili

$$0 \leq \left| x \cdot \sin^2\left(\frac{1}{x}\right) \right| =$$

$$= |x| \cdot \left| \sin^2\left(\frac{1}{x}\right) \right| =$$

$$= |x| \cdot \underbrace{\sin^2\left(\frac{1}{x}\right)}_{\leq 1} \leq |x|$$

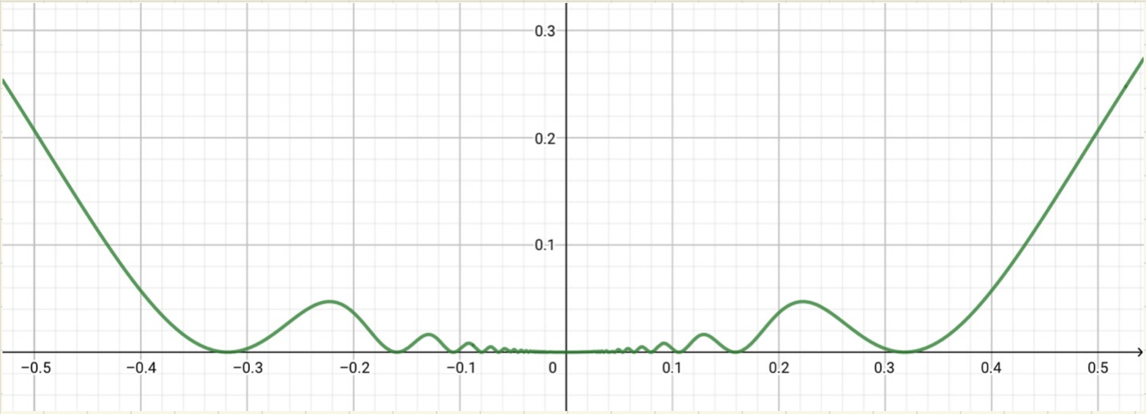
$$\sin^2\left(\frac{1}{x}\right) \leq 1$$

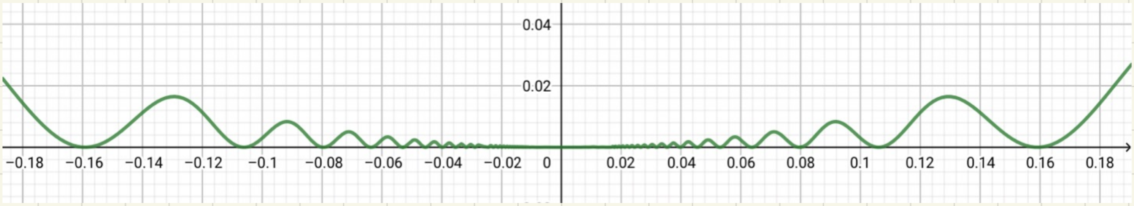
moltiplico per $|x| > 0$ ambo i membri

$$|x| \cdot \sin^2\left(\frac{1}{x}\right) \leq |x|$$

Tuttavia f oscilla per $x \to 0$, dunque non è monotona vicino a zero, quindi f' non ha segno:

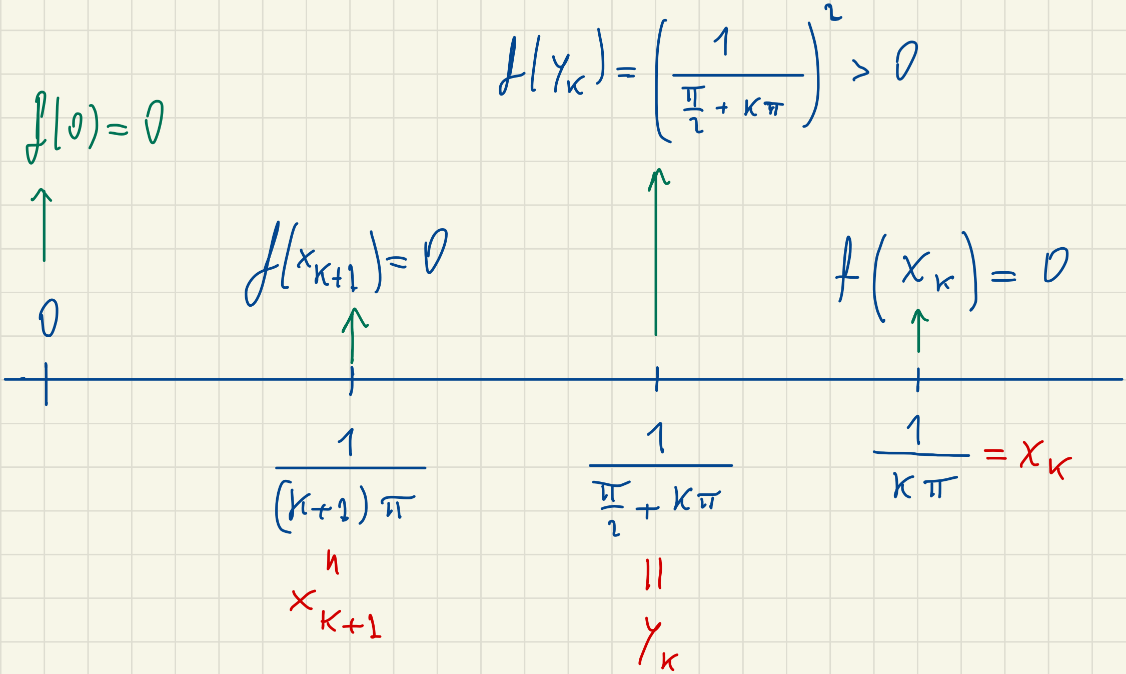
- ≤ 0 a sinistra di zero;
- ≥ 0 a destra di zero.





$$\begin{cases} x_k = \dfrac{1}{\pi k} \xrightarrow[k \to +\infty]{} 0^+ \\[2ex] f(x_k) = \left(\dfrac{1}{\pi k}\right)^2 \cdot \sin\left(\dfrac{1}{\frac{1}{\pi k}}\right)^2 = \\[2ex] \qquad = \left(\dfrac{1}{\pi k}\right)^2 \cdot \sin^2(k \cdot \pi) = 0 \end{cases}$$

$$\begin{cases} y_k = \dfrac{1}{\frac{\pi}{2} + k\pi} \xrightarrow[k \to +\infty]{} 0^+ \\[2ex] f(y_k) = \left(\dfrac{1}{\frac{\pi}{2} + k\pi}\right)^2 \cdot \boxed{\underbrace{\sin^2\left(\frac{\pi}{2} + k\pi\right)}_{=1}} = \\[2ex] \qquad = \left(\dfrac{1}{\frac{\pi}{2} + k\pi}\right)^2 > 0 \end{cases}$$



$\forall k \in \mathbb{N}$

GRAFICO QUALITATIVO DI UNA FUNZIONE f:

(1) Determinare il dominio naturale di f: $D(f)$

FACOLTATIVO:

(2) Individuazione di eventuali simmetrie:

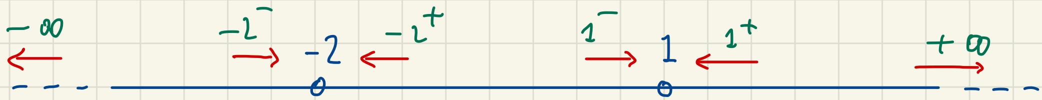
- f pari: $f(-x) = f(x)$
  ($\Longrightarrow$ il grafico è simmetrico rispetto l'asse delle y);

- f è dispari: $f(-x) = -f(x)$
  ($\Longrightarrow$ il grafico è simmetrico (centralmente) rispetto all'origine).

In tal caso, si può studiare f per $x \geq 0$.

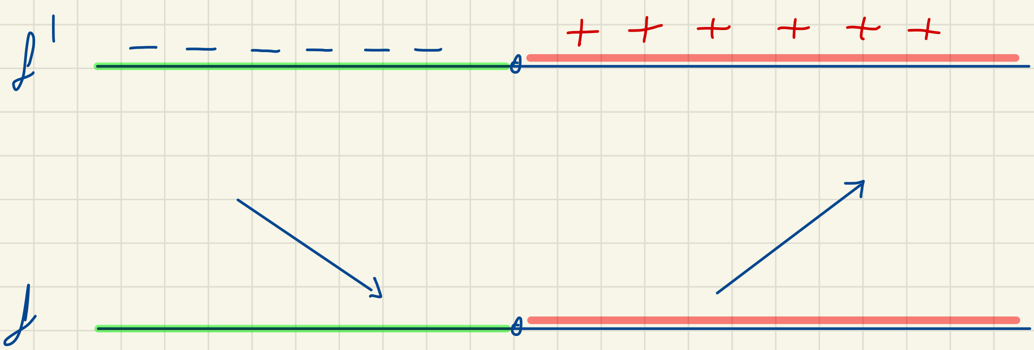
③ Calcolo dei limiti relativi alla frontiera del dominio di f. (calcolo degli asintoti vert. e orizz.)

Esempio: $D(f) = \mathbb{R} \setminus \{-2, 1\}$



④ Studio della monotonia di f attraverso l'analisi del segno della sua derivata prima f'.

Esempio:



4 NEL DETTAGLIO:

- calcolo della derivata (nei punti in cui derivabilità)
- $f'(x) = 0$
- studio del segno di f'
  $\Rightarrow$ monotonia di f
- determinazione degli eventuali punti di min e di max

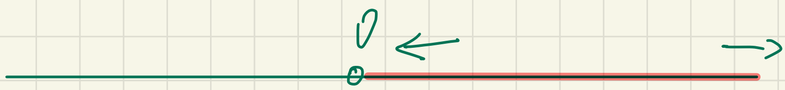
$$f'(\bar{x}) = 0$$

$$\downarrow$$

$$(\bar{x}, f(\bar{x}))$$

DSS :

$D(f) = \{ x \mid x > 0 \}$



$\lim_{x \to 0^+}$ $\qquad$ $\lim_{x \to +\infty}$

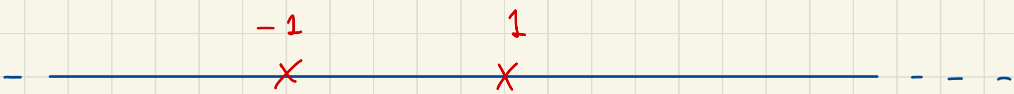
$\cancel{\lim_{x \to 0}}$

OSS.:

Se f è pari o dispari, il suo dominio $D(f)$ deve essere simmetrico rispetto l'origine.
Quindi, se $D(f)$ non è simmetrico $\Longrightarrow$ f non può essere né pari né dispari

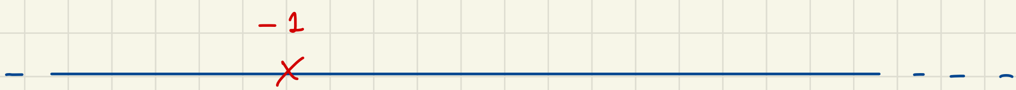
Esempio:

①

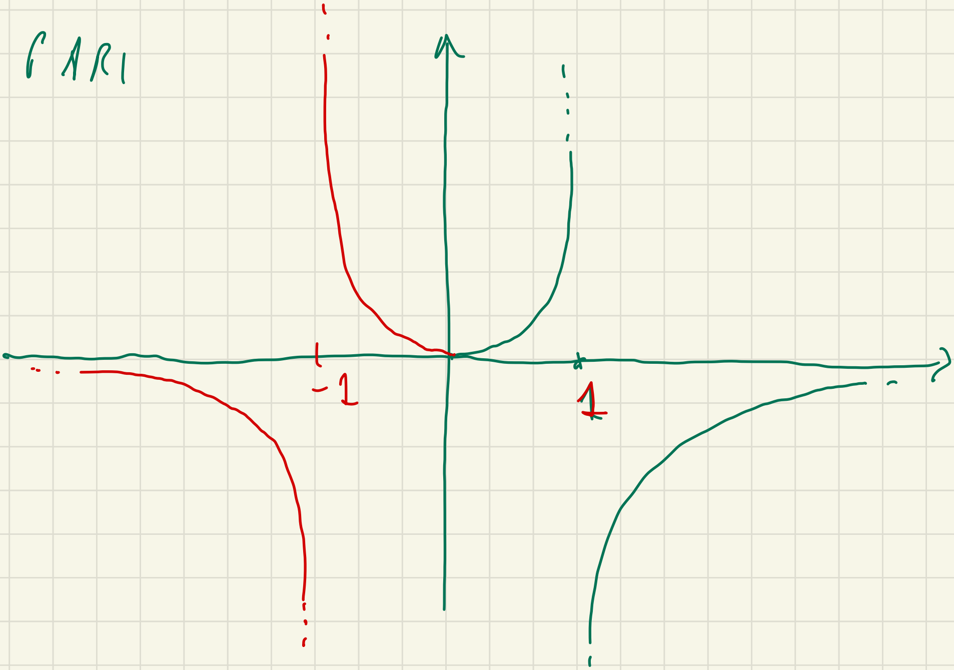


$D(f) = \mathbb{R} \setminus \{\pm 1\}$ $\longrightarrow$ f <u>PUÒ</u> essere pari o dispari

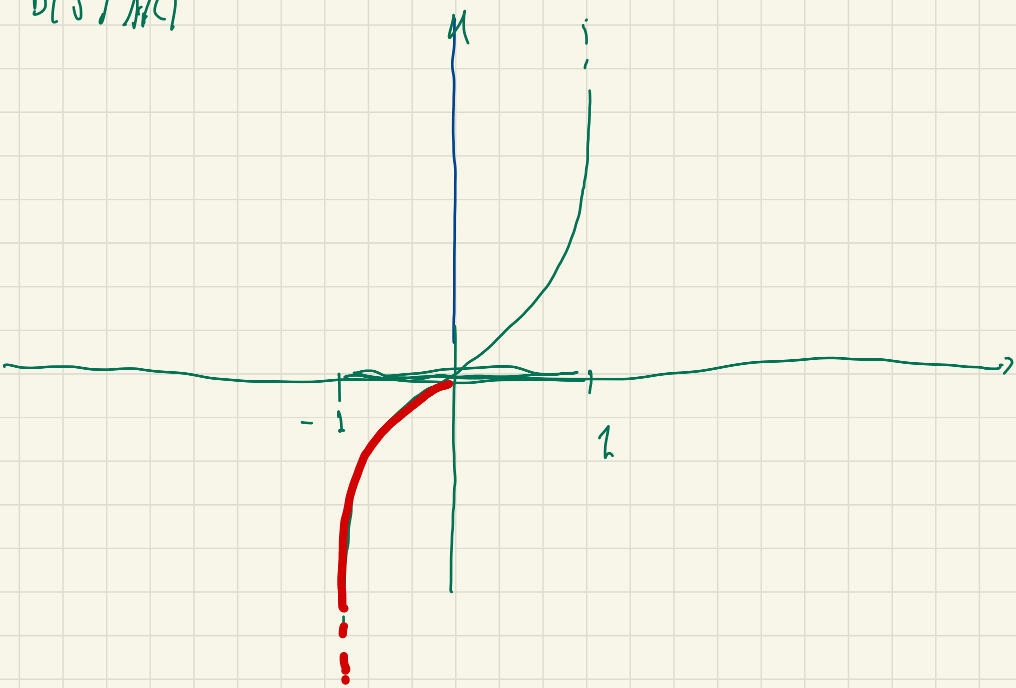
①



$D(f) = \mathbb{R} \setminus \{-1\}$ $\longrightarrow$ f <u>non</u> è pari e <u>non</u> è dispari f PARI



f DISPARI



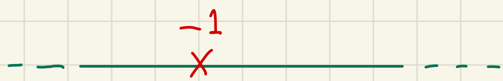
Esempio 1:

Disegnare il "grafico qualitativo" di f:

$$f(x) = e^{\frac{x^2-3}{x+1}}$$

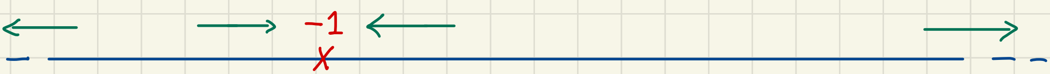
(1) $D(f) = \{ x \in \mathbb{R} \mid x+1 \neq 0 \} =$

$= \mathbb{R} \setminus \{-1\}$



$\Rightarrow f$ non è né pari né dispari.

(3)



$$\lim_{x \to -\infty} e^{\frac{x^2-3}{x+1}} = ?$$

$$\lim_{x\to-\infty} \frac{x^2-3}{x+1} = \lim_{x\to-\infty} \underbrace{\frac{x^2}{x}}_{=\,x\,\to\,-\infty}\cdot \underbrace{\frac{1-\frac{3}{x^2}}{1+\frac{1}{x}}}_{\to\,1} = -\infty$$

$$\longrightarrow \bullet \lim_{x\to-\infty} e^{\frac{x^2-3}{x+1}} = 0^+$$

$$\bullet \lim_{x\to+\infty} e^{\overbrace{\frac{x^2-3}{x+1}}^{\to\,+\infty}} = +\infty$$

$$\lim_{x\to-1^-} e^{\frac{x^2-3}{x+1}} = \,?$$

$$\lim_{x\to-1^-} \frac{\overbrace{x^2-3}^{\to\,-2}}{\underbrace{x+1}_{\to\,0^-}} = \lim_{x\to-1^-} \overbrace{(x^2-3)}^{-2}\cdot\overbrace{\frac{1}{x+1}}^{-\infty} = +\infty$$

$$\longrightarrow \bullet \lim_{x\to-1^-} e^{\frac{x^2-3}{x+1}} = +\infty$$

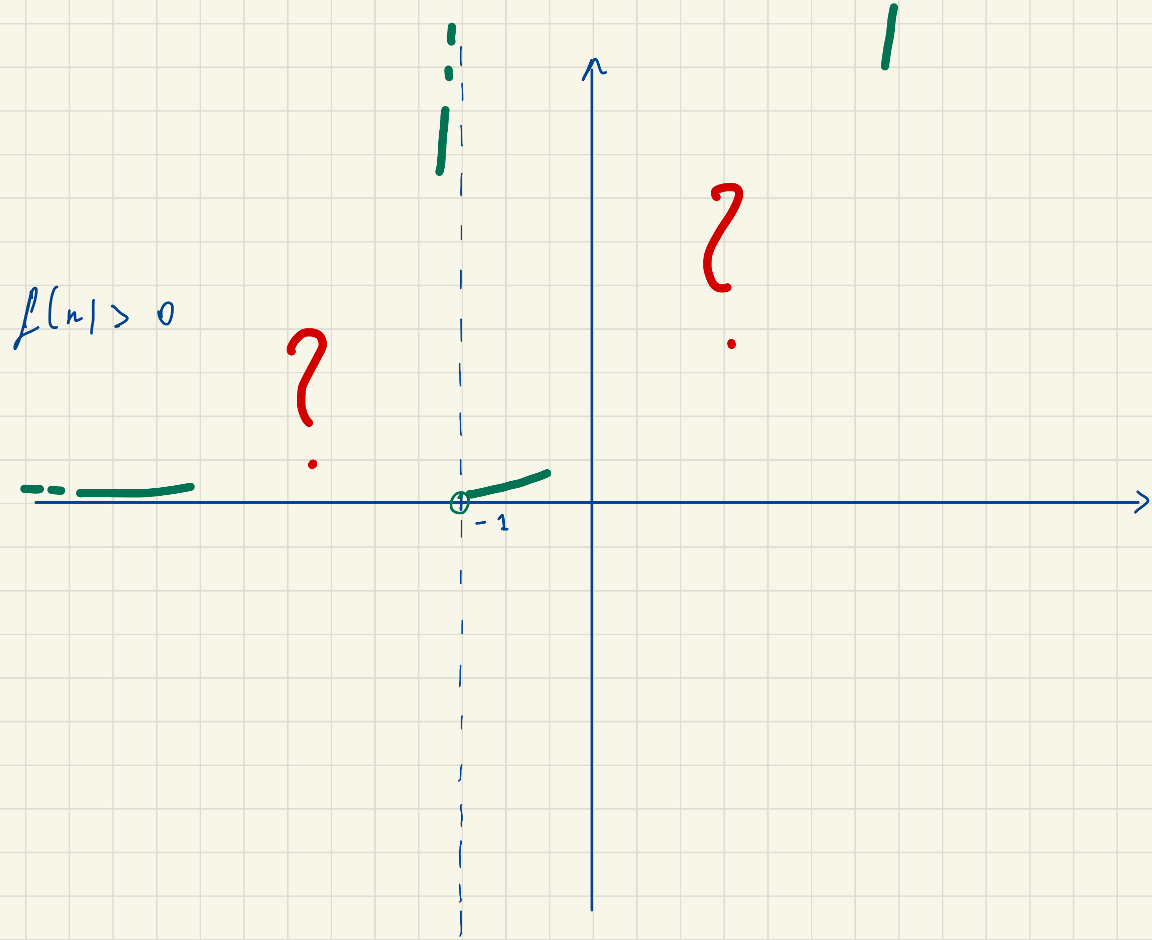
$x+1>0 \iff x>-1$

x		-1	
$x+1$	$-\;-\;-\;-\;-$		$+\;+\;+$

$$\lim_{x\to -1^+} e^{\frac{x^2-3}{x+1}} = ?$$

$$\lim_{x\to -1^+} \frac{\overbrace{x^2-3}^{\to -2}}{\underbrace{x+1}_{\to 0^+}} = \lim_{x\to -1^+} \overbrace{(x^2-3)}^{-2}\cdot \overbrace{\frac{1}{x+1}}^{+\infty} = -\infty$$

$$\longrightarrow \quad \bullet \quad \lim_{x\to -1^+} e^{\frac{x^2-3}{x+1}} = 0^+$$



4

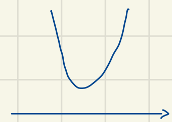
$$f(x) = e^{\frac{x^2-3}{x+1}}$$

$$f'(x) = e^{\frac{x^2-3}{x+1}} \cdot \frac{2x(x+1) - 1\cdot(x^2-3)}{(x+1)^2} =$$

$$= e^{\frac{x^2-3}{x+1}} \cdot \frac{2x^2+2x-x^2+3}{(x+1)^2} =$$

$$= \underset{\substack{\vee \\ 0}}{\boxed{\frac{e^{\frac{x^2-3}{x+1}}}{(x+1)^2}}} \cdot \left(\underline{x^2+2x+3} \right)$$

$$x^2+2x+3 = 0$$



$$\Delta = 4 - 12 < 0$$

$$\Longrightarrow \quad x^2+2x+3 > 0 \quad \forall x$$

$$f'(x) > 0 \quad \forall x \in D(f)$$

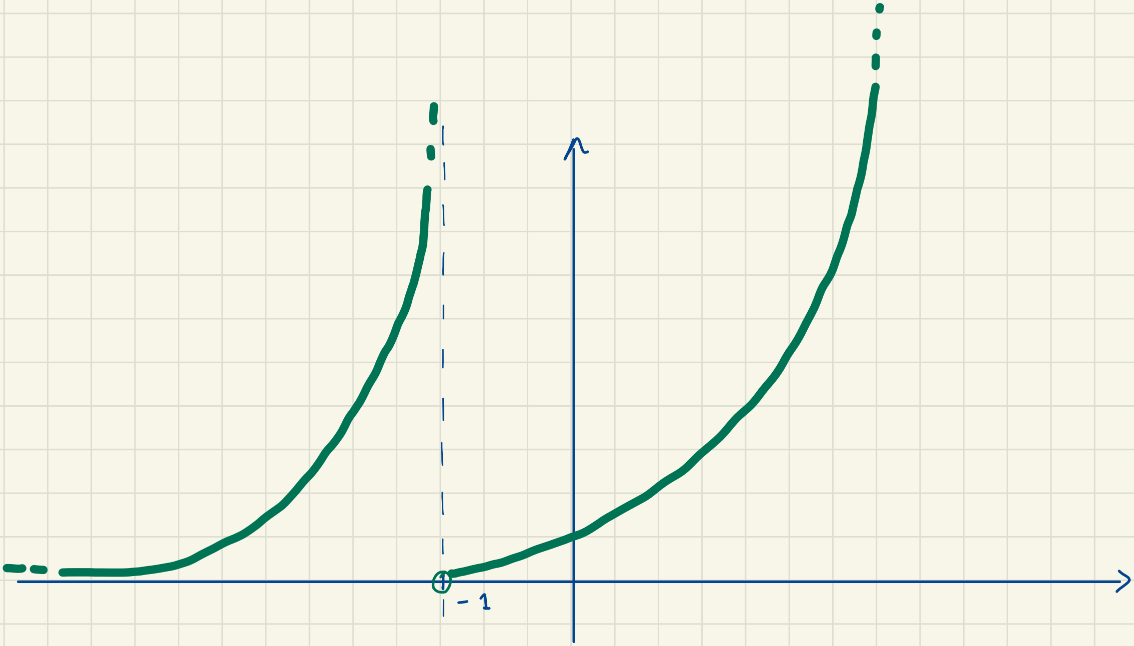
$\Longrightarrow$ f è (strett.) crescente

DOMANDE:

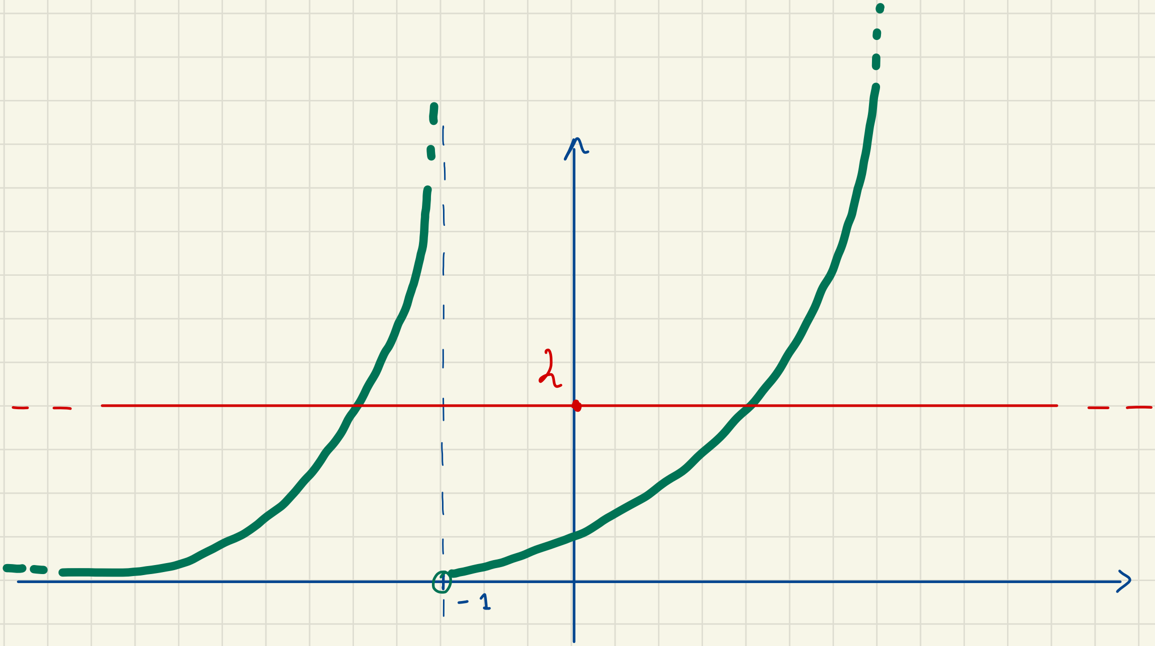
1. Calcolare $\operatorname{Im} f$

2. Stabilire QUANTE soluzioni ha l'equazione:

$$f(x) = \lambda \qquad (\lambda \in \mathbb{R})$$



$$\operatorname{Im} f = \{ y \in \mathbb{R} \mid y > 0 \}$$

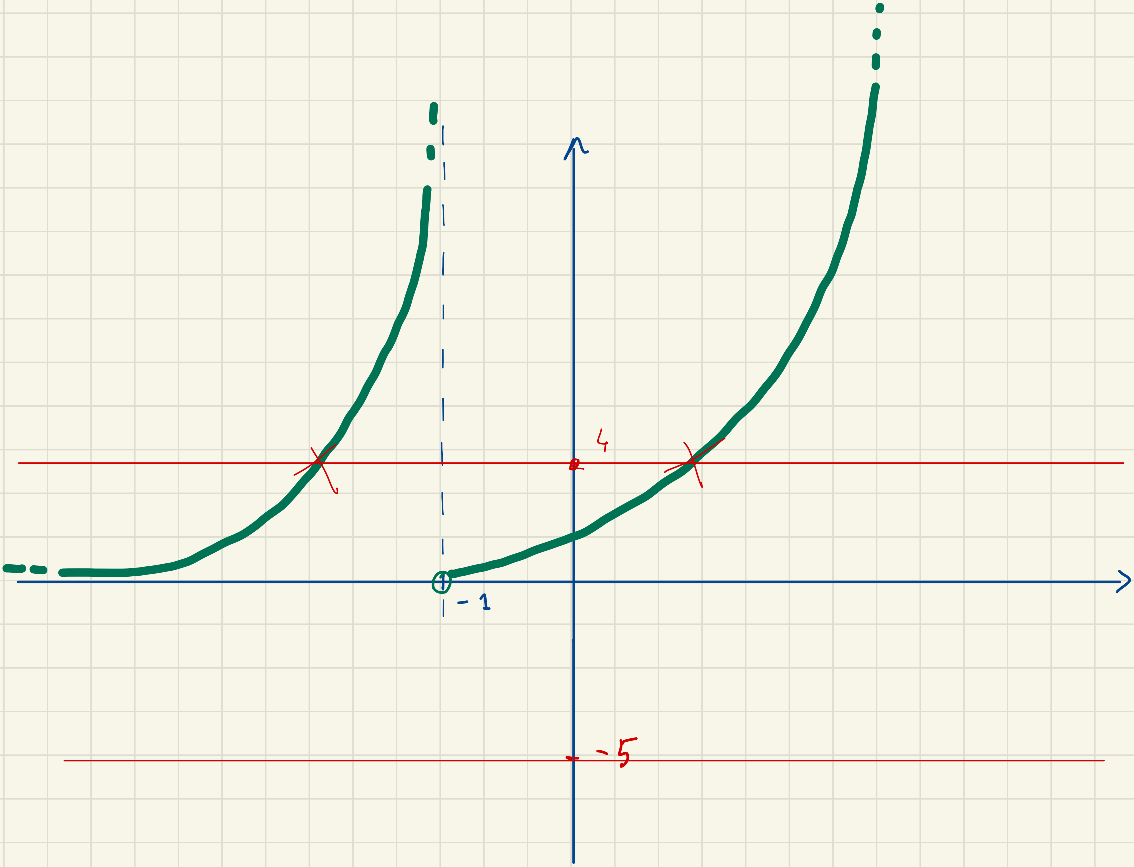


$$f(x) = \lambda$$

$$\downarrow$$

$$\begin{cases} y = f(x) \\ y = \lambda \end{cases}$$

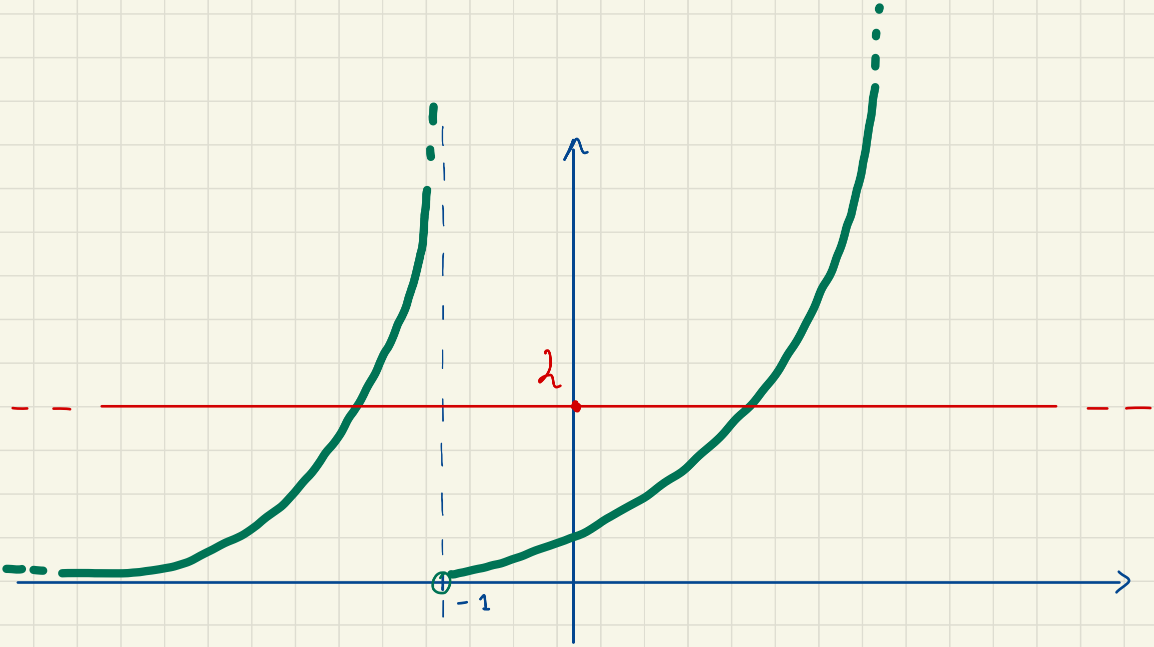
$$S = \{ x \in D(f) \mid f(x) = \lambda \}$$



$f(x) = -5 \longrightarrow \# S = 0$

$f(x) = 4 \longrightarrow \# S = 2$

$f(x) = 0 \longrightarrow \# S = 0$



(2) Per quali valori di $\lambda \in \mathbb{R}$ l'equazione $f(x) = \lambda$ ha 2 soluzioni (distinte)?

$\underline{\lambda > 0}$

Un quesito: $\lambda \in \mathbb{R}$

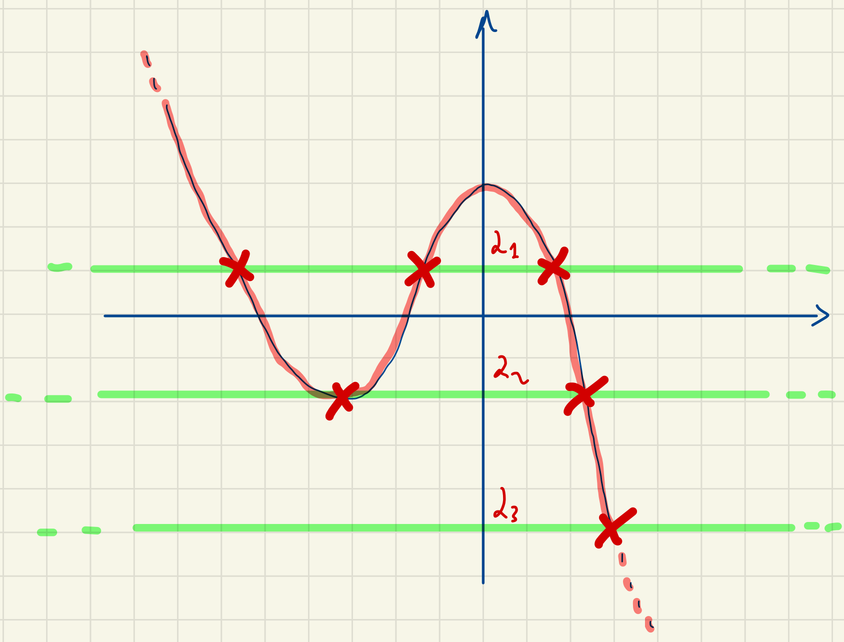
$$f(x) = \lambda \qquad (*)$$

Quante soluzioni ha l'equazione $(*)$?

$$f(x) = \lambda$$ (lambda)

$$\Updownarrow$$

$$\begin{cases} y = f(x) \\ y = \lambda \end{cases}$$



Esercizio 2:

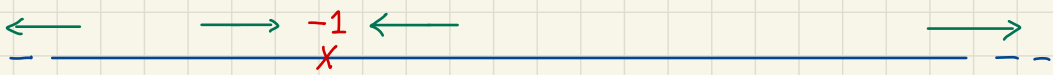
$$f(x) = e^{\frac{x^2+3}{x+1}}$$

1. $D(f) = \{ x \in \mathbb{R} \mid x+1 \neq 0 \} =$

$$= \mathbb{R} \setminus \{-1\}$$

$\Rightarrow$ f non è né pari né dispari.

3.



$$\lim_{x \to -\infty} e^{\frac{x^2+3}{x+1}} = ?$$

$$\lim_{x\to-\infty} \frac{x^2+3}{x+1} = \lim_{x\to-\infty} \underbrace{\frac{x^2}{x}}_{\substack{= x \\ \downarrow \\ -\infty}} \cdot \underbrace{\frac{1+\frac{3}{x^2}}{1+\frac{1}{x}}}_{\substack{\downarrow \\ 1}} = -\infty$$

$\longrightarrow$ • $\displaystyle\lim_{x\to-\infty} e^{\frac{x^2+3}{x+1}} = 0^+$

• $\displaystyle\lim_{x\to+\infty} e^{\frac{x^2+3}{x+1}} = +\infty \qquad \frac{x^2+3}{x+1} \to +\infty$

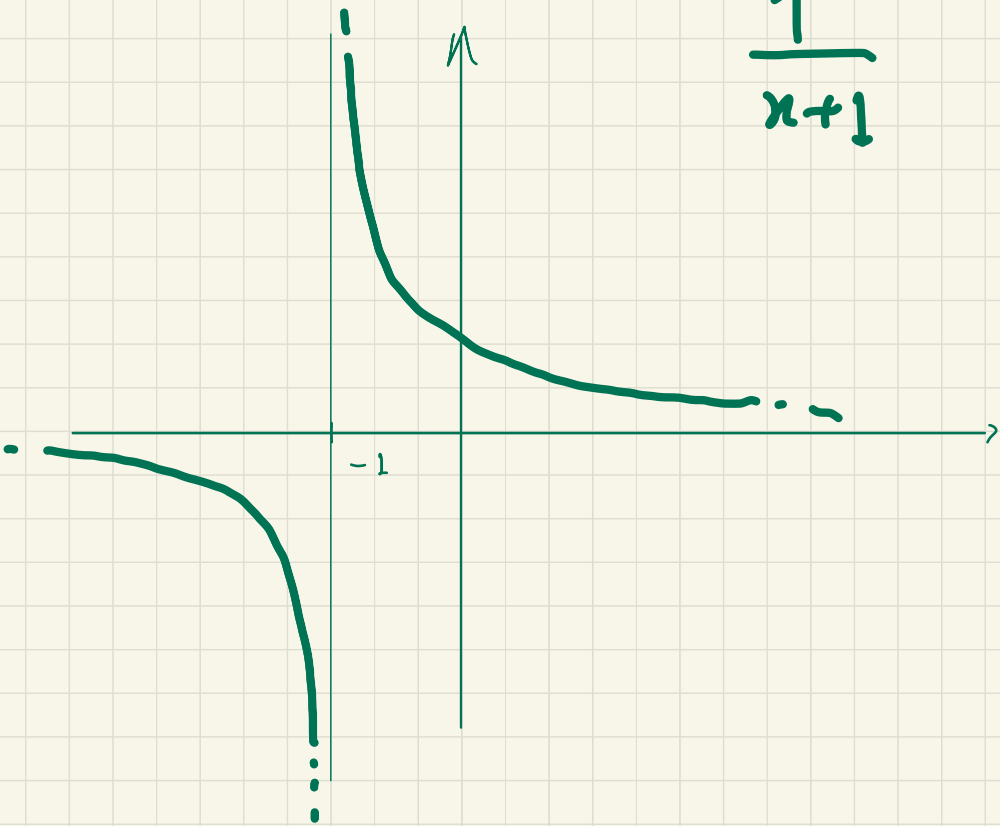
$$\lim_{x\to-1^-} e^{\frac{x^2+3}{x+1}} = ?$$

$$\lim_{x\to-1^-} \frac{\overbrace{x^2+3}^{\to 4}}{\underbrace{x+1}_{\to 0^-}} = \lim_{x\to-1^-} \overbrace{(x^2+3)}^{4} \cdot \overbrace{\frac{1}{x+1}}^{-\infty} = -\infty$$

$\longrightarrow$ • $\displaystyle\lim_{x\to-1^-} e^{\frac{x^2+3}{x+1}} = 0^+$

$-\ -\ -\ \ -1\ \ +\ +\ +$

$$\frac{1}{x+1}$$



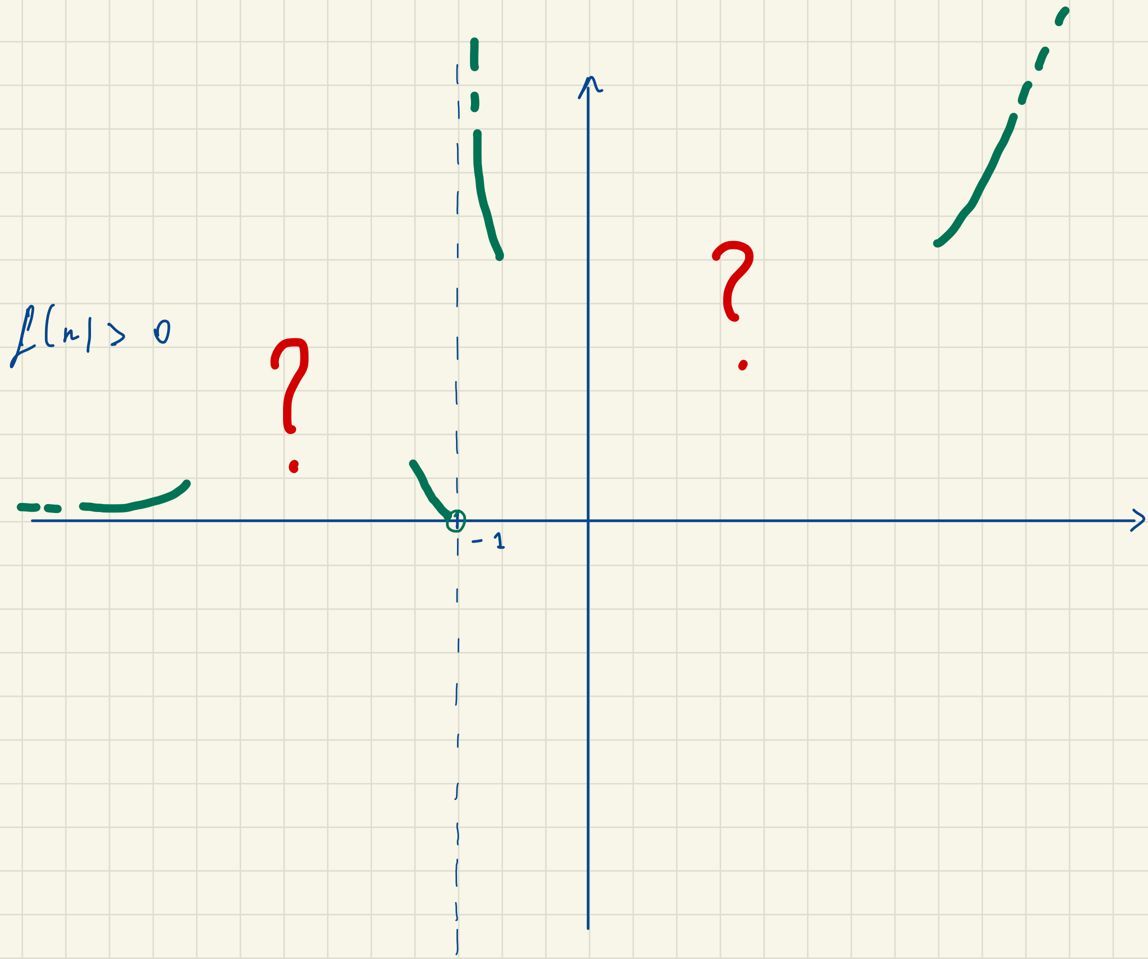
$x+1$	$-$	-1	$+$

$$\lim_{x \to -1^+} e^{\frac{x^2+3}{x+1}} = ?$$

$$\lim_{x \to -1^+} \frac{x^2+3}{x+1} = \lim_{x \to -1^+} \underbrace{(x^2+3)}_{\to 4} \cdot \underbrace{\frac{1}{x+1}}_{\to +\infty} = +\infty$$

$x^2+3 \to 4$, $x+1 \to 0^+$

$$\longrightarrow \quad \bullet \lim_{x \to -1^+} e^{\frac{x^2+3}{x+1}} = +\infty$$



4

$$f(x) = e^{\frac{x^2+3}{x+1}}$$

$$f'(x) = e^{\frac{x^2+3}{x+1}} \cdot \frac{2x(x+1) - 1 \cdot (x^2+3)}{(x+1)^2} =$$

$$= e^{\frac{x^2+3}{x+1}} \cdot \frac{2x^2+2x-x^2-3}{(x+1)^2} =$$

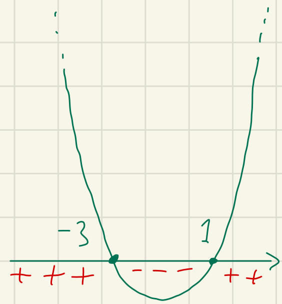
$$= \boxed{\frac{e^{\frac{x^2+3}{x+1}}}{(x+1)^2}} \cdot \left(\underline{x^2+2x-3}\right)$$

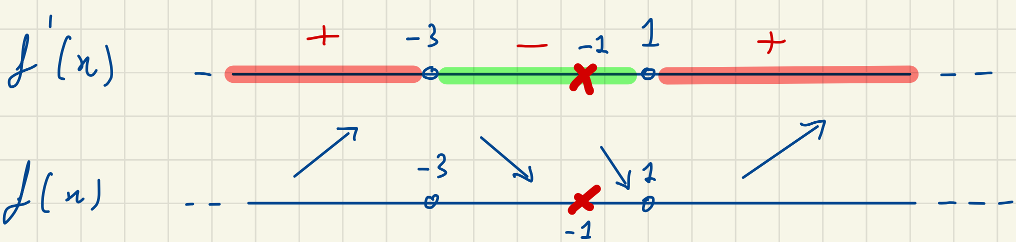
$\vee$
0

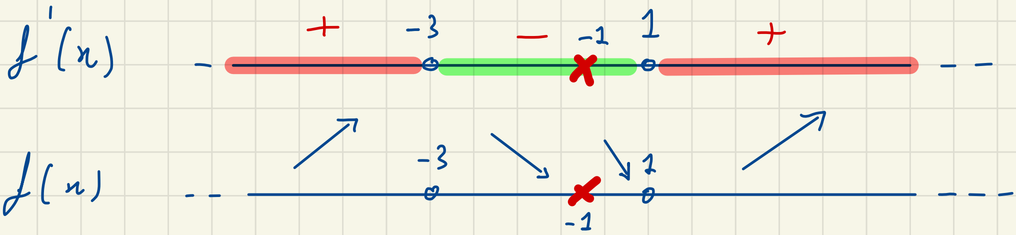
$$x^2+2x-3=0$$

$$\Delta = 4+12 = 16$$

$$x_{1,2} = \frac{-2 \pm 4}{2} = \begin{cases} -3 \\ 1 \end{cases}$$







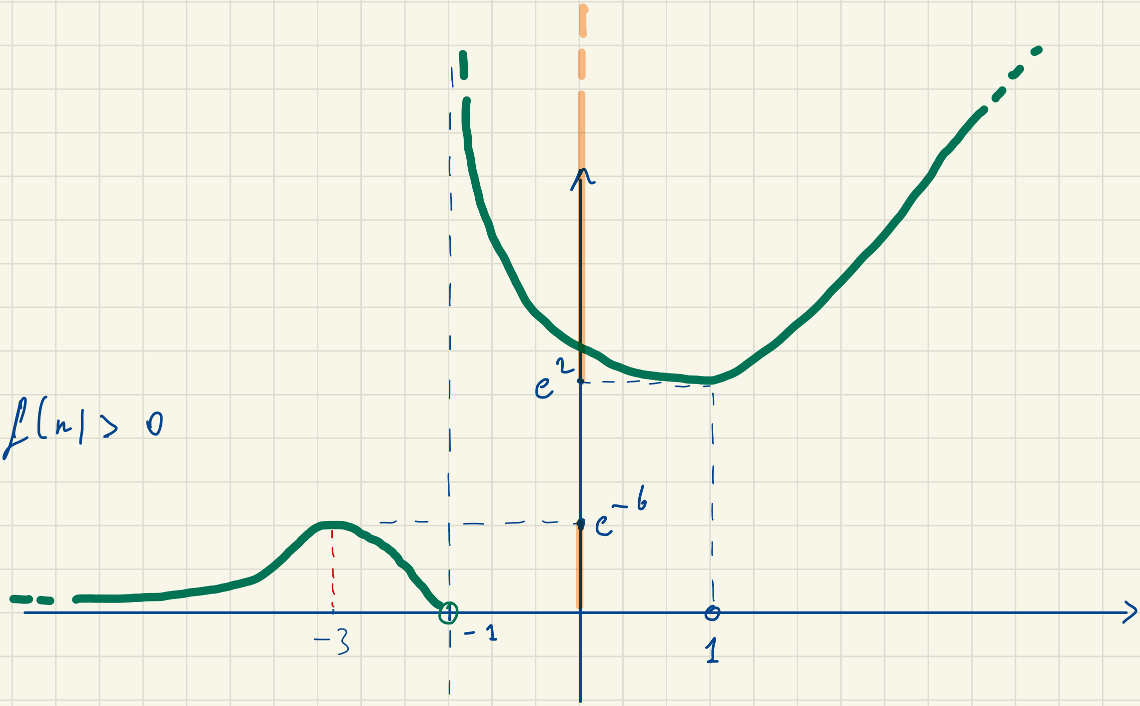
$x = -3$ p. di max relativo

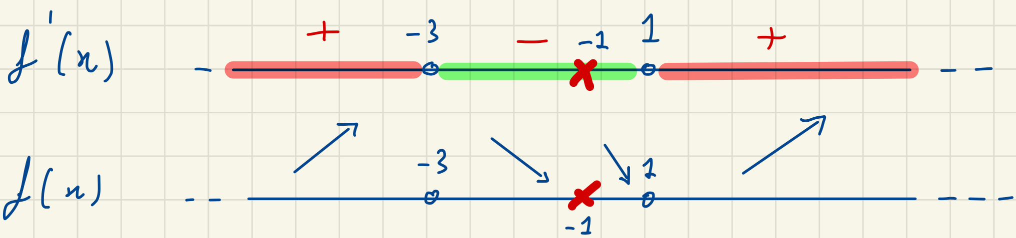
$$f(-3) = e^{\frac{x^2+3}{x+1}}\Bigg|_{x=-3} = e^{\frac{12}{-2}} = e^{-6}$$

$x = 1$ p. di min relativo

$$f(1) = e^{\frac{x^2+3}{x+1}}\Bigg|_{x=1} = e^{\frac{4}{2}} = e^{2}$$

$f(x) > 0$





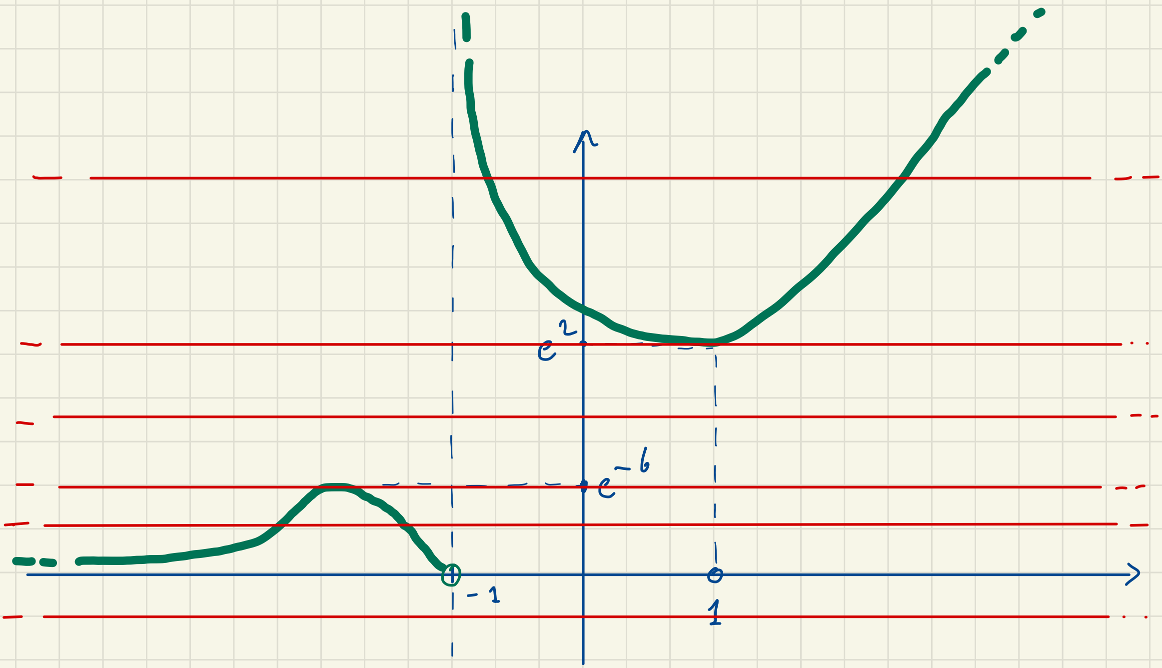
$$\mathrm{Im}\, f = \,]0, e^{-6}] \cup [e^2, +\infty[$$

\# S

Discutere il numero di soluzioni dell'equazione: (senza risolverla)

$$e^{\frac{x^2+3}{x+1}} = \lambda$$

al variare di $\lambda \in \mathbb{R}$.

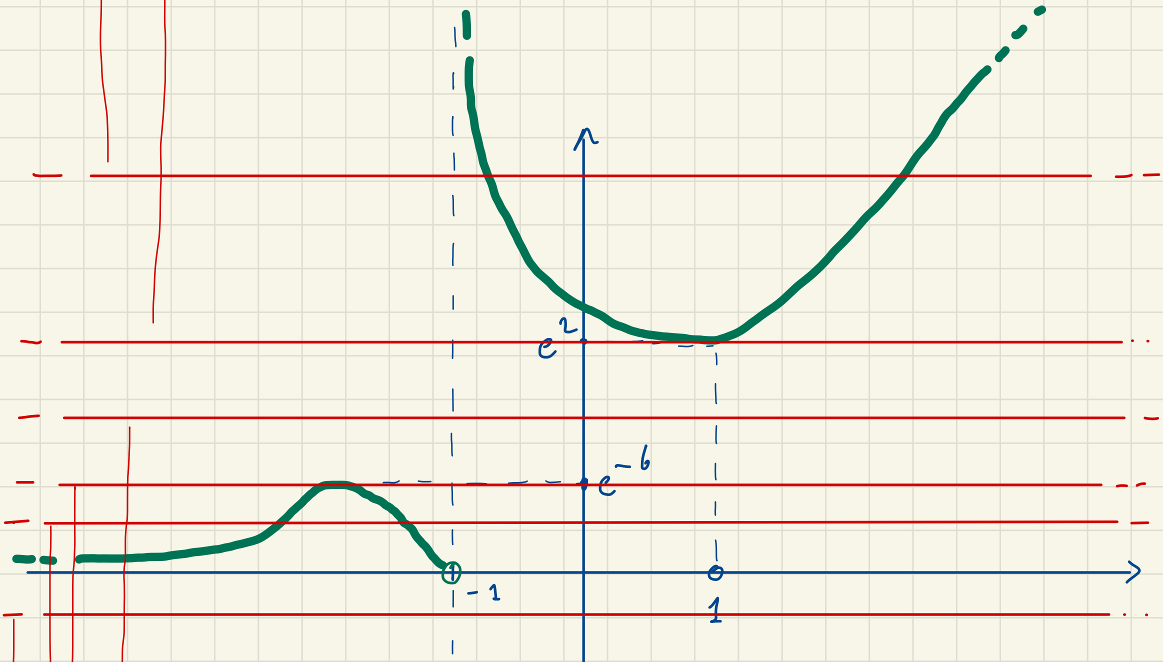


$$S = \{ x \in D(f) \mid f(x) = \lambda \}$$

$\lambda > e^2 \qquad \# S = 2$

$\lambda = e^2 \qquad \# S = 1$

$\boxed{\# S = \text{il num. di elementi di } S}$



$e^{-6} < \lambda < e^2 \qquad \# S = 0$

$\lambda = e^{-6} \qquad \# S = 1$

$0 < \lambda < e^{-6} \qquad \# S = 2$

$\lambda \leq 0 \qquad \# S = 0$

$f(x) = \lambda$

1. non ha soluzione:
   se $\lambda \in]-\infty, 0] \cup]e^{-6}, e^{2}[$
   $(\lambda \leq 0$ oppure $e^{-6} < \lambda < e^{2})$

2. ha 1 soluzione:
   se $\lambda = e^{-6}$ o $\lambda = e^{2}$

3. ha 2 soluzioni:
   se $\lambda \in]0, e^{-6}[\cup]e^{2}, +\infty[$
   $(0 < \lambda < e^{-6}$ oppure $\lambda > e^{2})$

ESERCIZIO 3:

$$f(x) = e^{\frac{x^2+1}{x^2-1}}$$

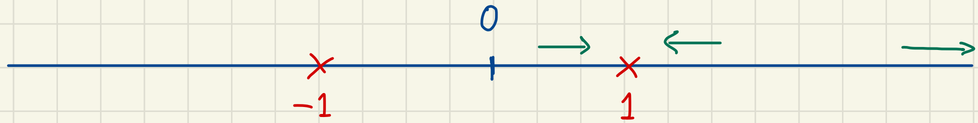
1 $D(f) = \{x \in \mathbb{R} \mid x^2 - 1 \neq 0\}$

$= \mathbb{R} \setminus \{\pm 1\}$

2 $f(-x) = e^{\frac{(-x)^2+1}{(-x)^2-1}} = e^{\frac{x^2+1}{x^2-1}} = f(x)$

$\Rightarrow$ f è pari $\Rightarrow$ il grafico di f è simmetrico rispetto l'asse delle y.

Possiamo studiare f solo per $x \geq 0$



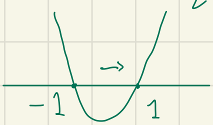
$$f(0) = e^{-1} = \frac{1}{e}$$

$$\lim_{x \to 1^-} e^{\frac{x^2+1}{x^2-1}} = 0^+$$

$$\left(\lim_{x \to 1^-} \frac{\overset{\to 2}{x^2+1}}{x^2-1} = -\infty \right)$$

1^+ $\quad +\infty$

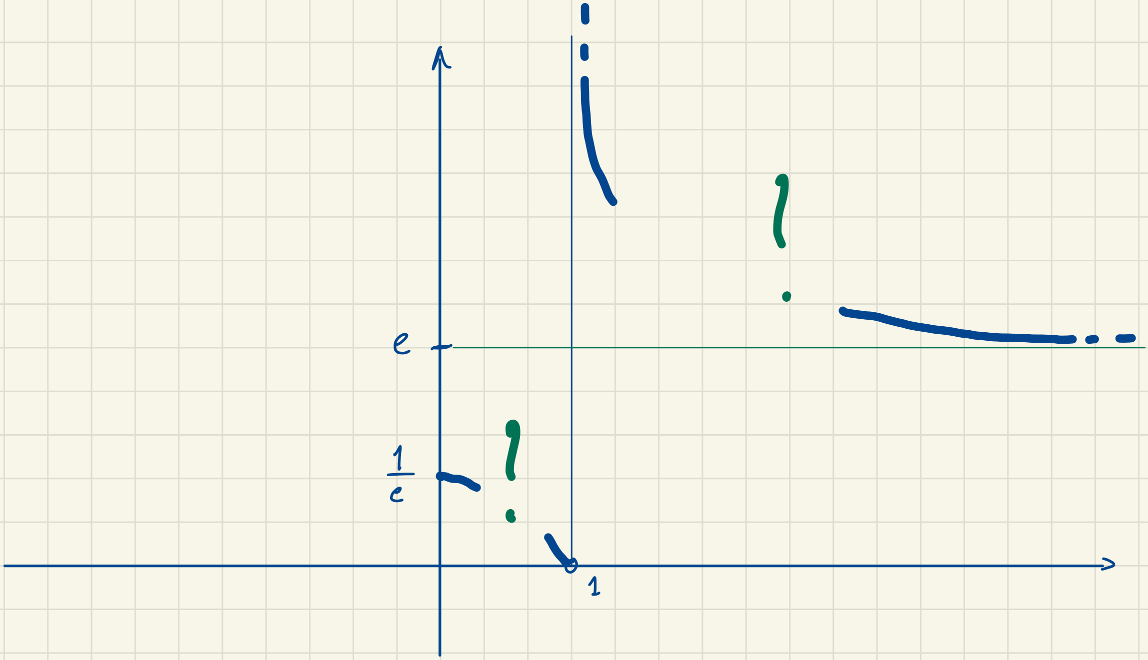
0^-



0^+

$$\lim_{x \to 1^+} e^{\frac{x^2+1}{x^2-1}} = +\infty$$

$$\lim_{x \to +\infty} e^{\frac{x^2+1}{x^2-1}} = \lim_{x \to +\infty} e^{\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}} = e$$



4

$$f(x) = e^{\frac{x^2+1}{x^2-1}}$$

$$f'(x) = e^{\frac{x^2+1}{x^2-1}} \cdot \frac{2x(x^2-1) - 2x(x^2+1)}{(x^2-1)^2} =$$

$$= e^{\frac{x^2+1}{x^2-1}} \cdot \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2-1)^2} =$$

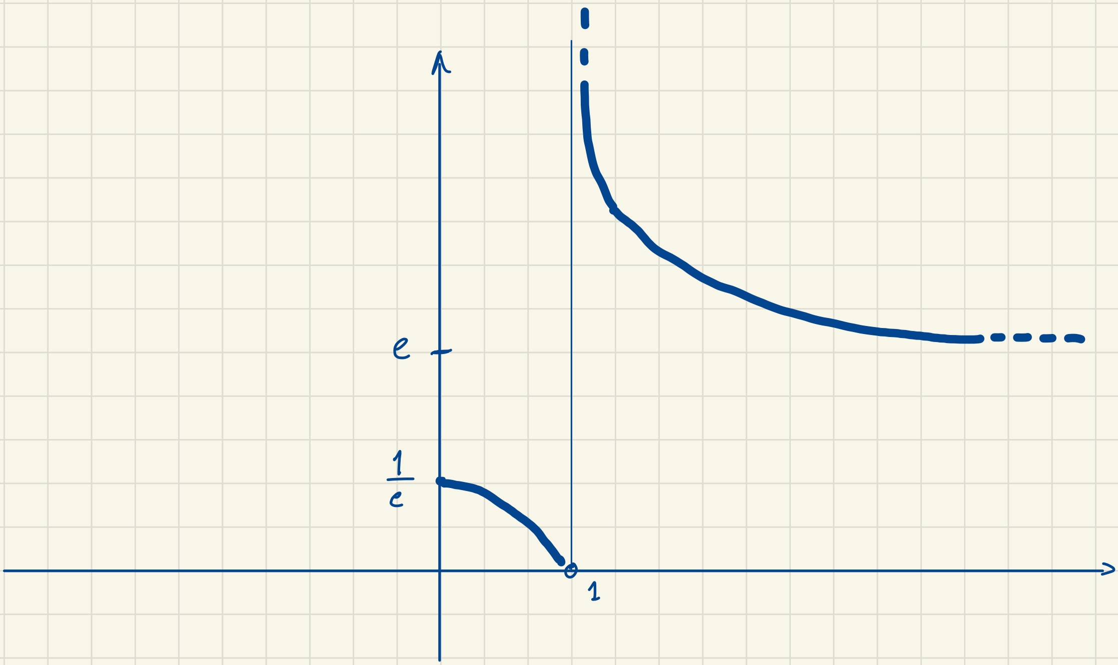
$$= \frac{e^{\frac{x^2+1}{x^2-1}}}{(x^2-1)^2} \cdot (-4x) \leq 0 \quad \text{se} \quad x \geq 0$$

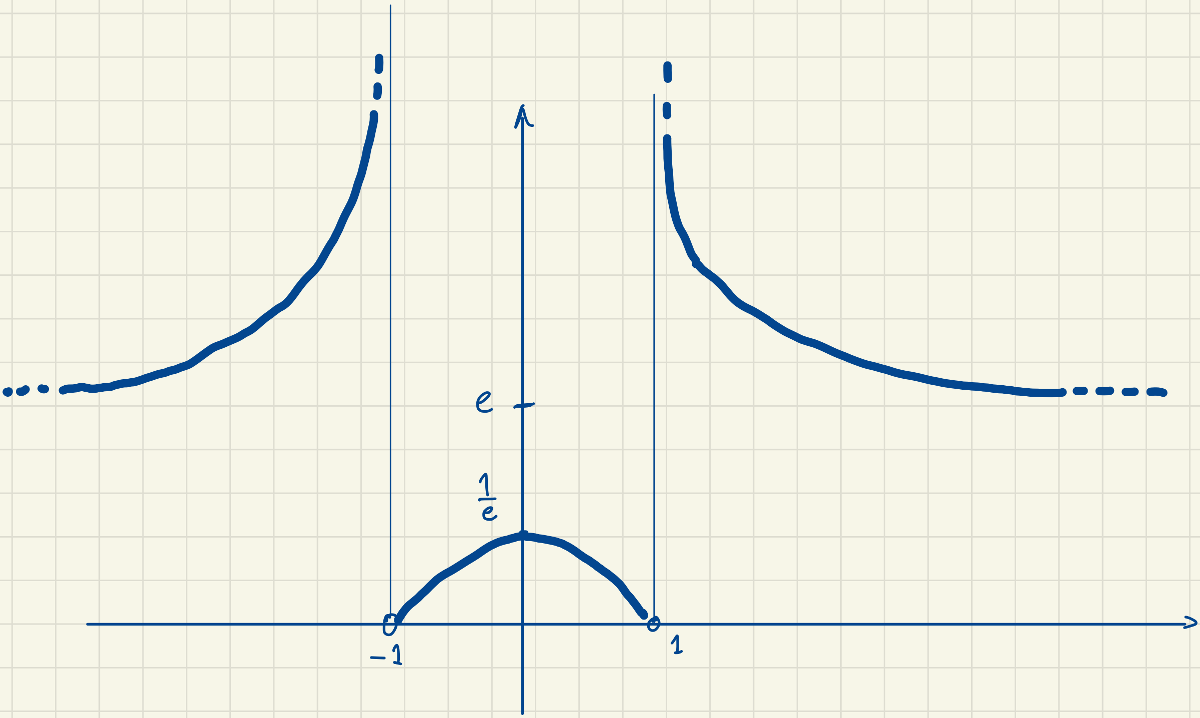
$$< 0 \quad \text{se} \quad x > 0$$

$f'(0) = 0$

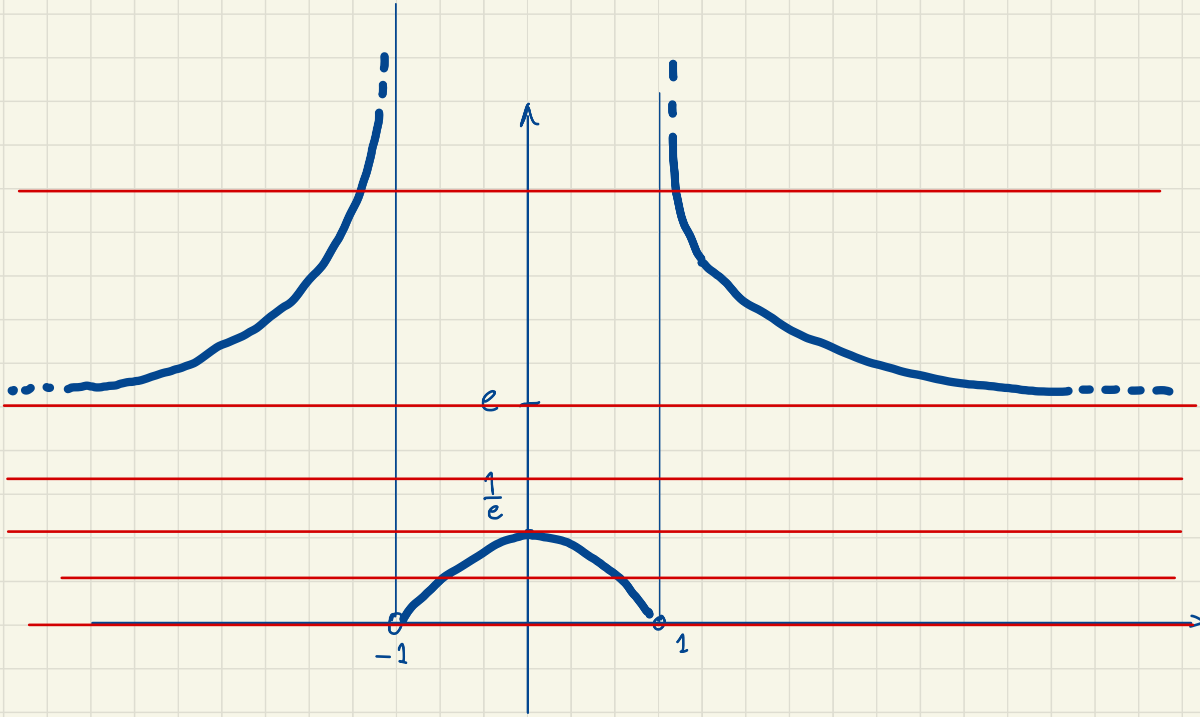
$f'(x) < 0$ se $x > 0$ $(x \neq 1)$

$\Rightarrow$ f è (strett.) decrescente se $x > 0$ $(x \neq 1)$





$$\mathrm{Im}\, f = \left]0, \frac{1}{e}\right] \cup \left]e, +\infty\right[$$



$$e < \lambda \longrightarrow \# S = 2$$

$$\frac{1}{e} < \lambda \le e \longrightarrow \# S = 0$$

$$\lambda = \frac{1}{e} \longrightarrow \# S = 1$$

$$0 < \lambda < \frac{1}{e} \longrightarrow \# S = 2$$

$$\lambda \le 0 \longrightarrow \# S = 0$$

$f(x) = \alpha$ ha:

1) $\# S = 0$ se $\alpha \in]-\infty, 0] \cup [\frac{1}{e}, e]$

2) $\# S = 1$ se $\alpha = \frac{1}{e}$

3) $\# S = 2$ se $\alpha \in]0, \frac{1}{e}[\cup]e, +\infty[$