8. Novembre. 2021

Esempio 3

$$f(x) = x^n \qquad (n \in \mathbb{N}) \qquad (n > 2)$$

$$x_0 \in \mathbb{R}$$

$$\overbrace{\boxed{\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}}}^{= f'(x_0)} =$$

$$= \lim_{h \to 0} \boxed{\frac{(x_0+h)^n - x_0^n}{h}}$$

$$\Bigg((x_0 + h)^n \underset{\text{binomio di Newton}}{=} \sum_{j=0}^{n} \binom{n}{j} x_0^{n-j} h^j =$$

$$= x_0^n + \underbrace{\binom{n}{1}}_{= n} x_0^{n-1} \cdot h +$$

$$+ \sum_{j=2}^{n} \binom{n}{j} x_0^{n-j} h^j \Bigg)$$

$$= \lim_{h\to 0} \boxed{\frac{(x_0+h)^n - x_0^n}{h}} =$$

$$= \lim_{h\to 0} \frac{\cancel{x_0^n} + n\, x_0^{n-1}\cdot h + \sum_{j=2}^{n} \binom{n}{j} x_0^{n-j} h^j \cancel{- x_0^n}}{h} =$$

$$= \lim_{h\to 0} \left(n\cdot x_0^{n-1} + \sum_{j=2}^{n} \binom{n}{j} x_0^{n-j} \underbrace{h^{j-1}}_{\xrightarrow{h\to 0}\ 0} \right)$$

$j-1 \geq 1$

$$= n\cdot x_0^{n-1}$$

$$D\, x^n = n\, x^{n-1} \qquad (n \in \mathbb{N})$$

Esercizio 2 :

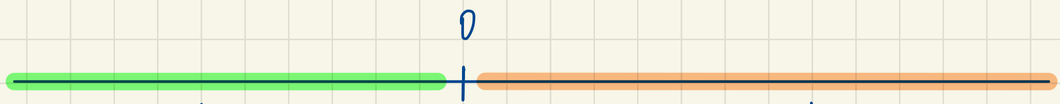
$a \in \mathbb{R}$

$f(x) = a \cdot x^n$ è derivabile

e $f'(x) = a \cdot n \cdot x^{n-1}$

Esempio 4:

$$f(x) = |x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$$



$\downarrow x < 0$

$f(x) \equiv -x$

$\downarrow x > 0$

$f(x) \equiv x$

$\Longrightarrow f$ è derivabile in $\mathbb{R} \setminus \{0\}$

$\bar{x} = 0$:

$$f'_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} =$$

$x < 0$

$$= \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} =$$

$$= \lim_{x \to 0^-} -1 = -1$$

$\Downarrow$

$x \neq 0$

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \quad x > 0$$

$$= \lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} =$$

$$= \lim_{x \to 0^+} +1 = +1$$

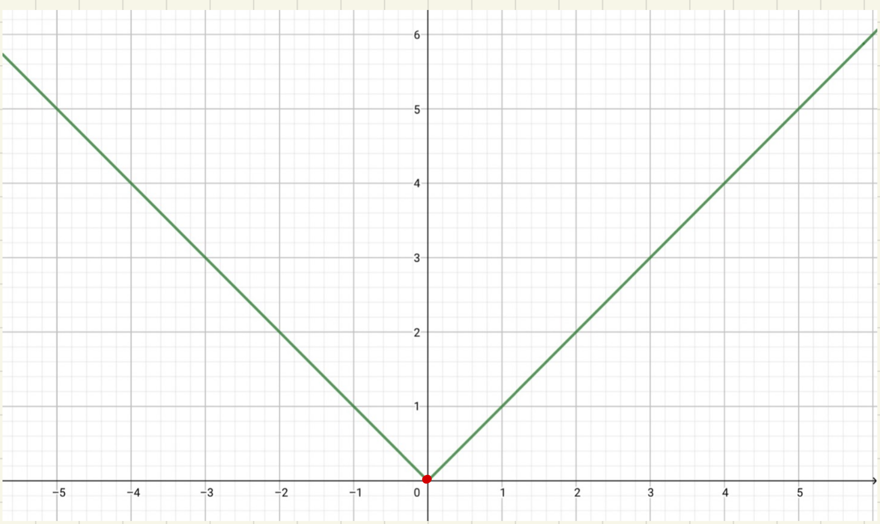
$\Downarrow$

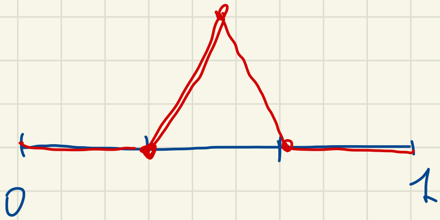
$x \neq 0$

$$\Longrightarrow \quad f'_-(0) = -1 \qquad f'_+(0) = 1$$

$$\neq$$

$\Longrightarrow$ f <u>non</u> è derivabile in $\bar{x} = 0$





0
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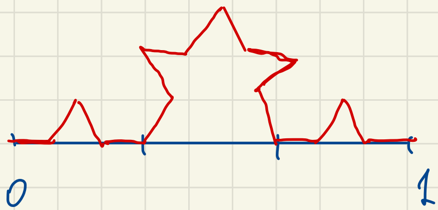


0
1

Attenzione: la presenza del valore assoluto all'interno di una funzione **non** è di per sé causa di NON-derivabilità.

Esempio:

$$f(x) = x|x| = \begin{cases} x^2 & \text{se } x \geq 0 \\ -x^2 & \text{se } x < 0 \end{cases}$$

Come prima, f è derivabile su $\mathbb{R} \setminus \{0\}$, poiché in tale regione f è un monomio (che si è visto essere derivabile).

Vediamo $\bar{x} = 0$:

$$f_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x} =$$

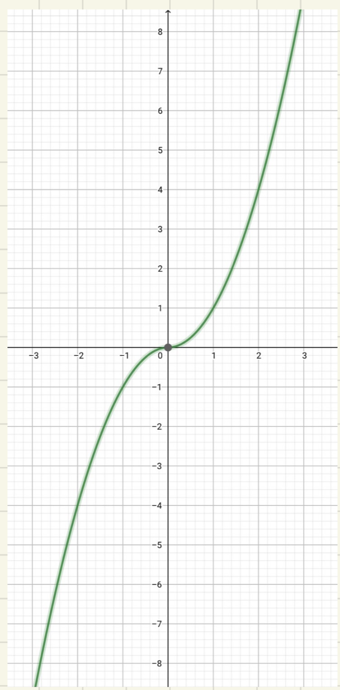
$$= \lim_{x \to 0^-} \frac{-x^2}{x} = \lim_{x \to 0^-} -x = 0$$

$$f_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} =$$

$$= \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} x = 0$$

$$\Longrightarrow \quad f_+(0) = 0 = f_-(0)$$

$\Longrightarrow$ f è derivabile in $\bar{x} = 0$: $f'(0) = 0$



Prop.: $f : I \longrightarrow \mathbb{R}$

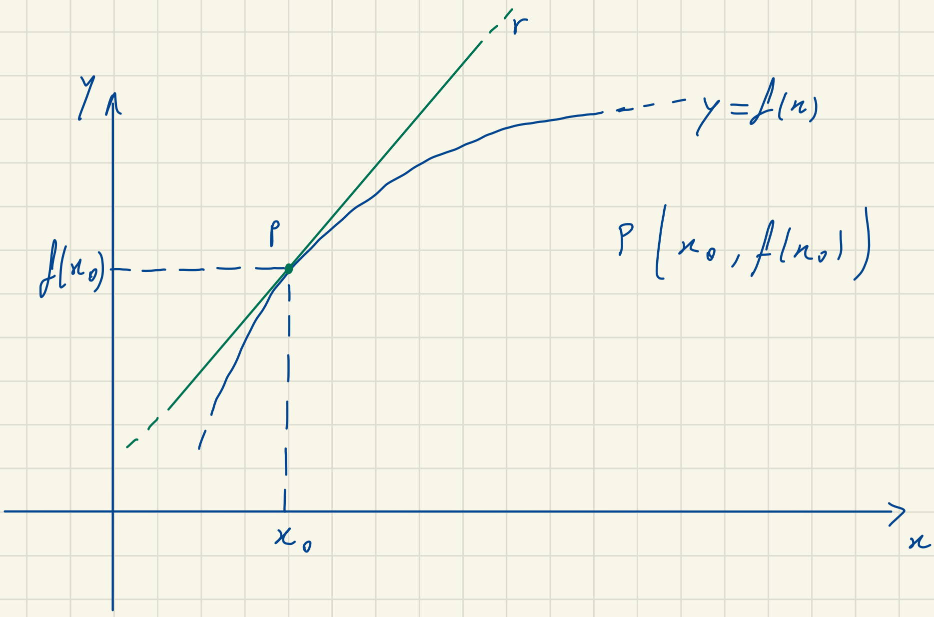
Se f è derivabile in $x_0 \in \mathring{I}$

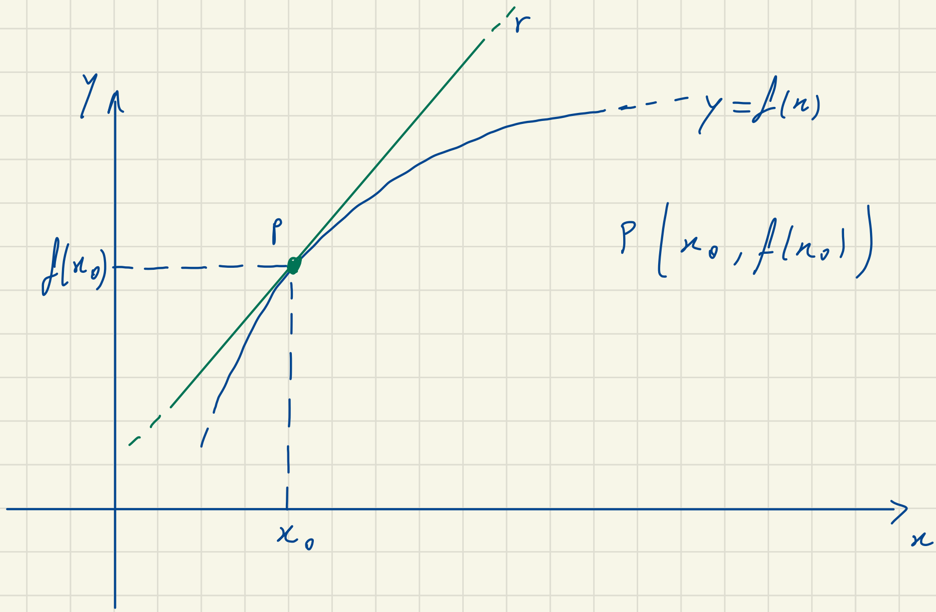
allora esiste la retta tangente

al grafico di f in $x = x_0$ ed

ha equazione:

$$r: \quad y = f'(x_0)(x - x_0) + f(x_0)$$





$$y = m(x - x_0) + f(x_0)$$

$$\boxed{y = f'(x_0)(x - x_0) + f(x_0)}$$

Esempio:

$$f(x) = \frac{1}{3} x^3$$

Calcoliamo la retta tangente r al grafico di f in $\bar{x} = 1$.

$$f'(x) = \frac{1}{3} \cdot 3x^2 = x^2 \Longrightarrow f'(1) = 1$$

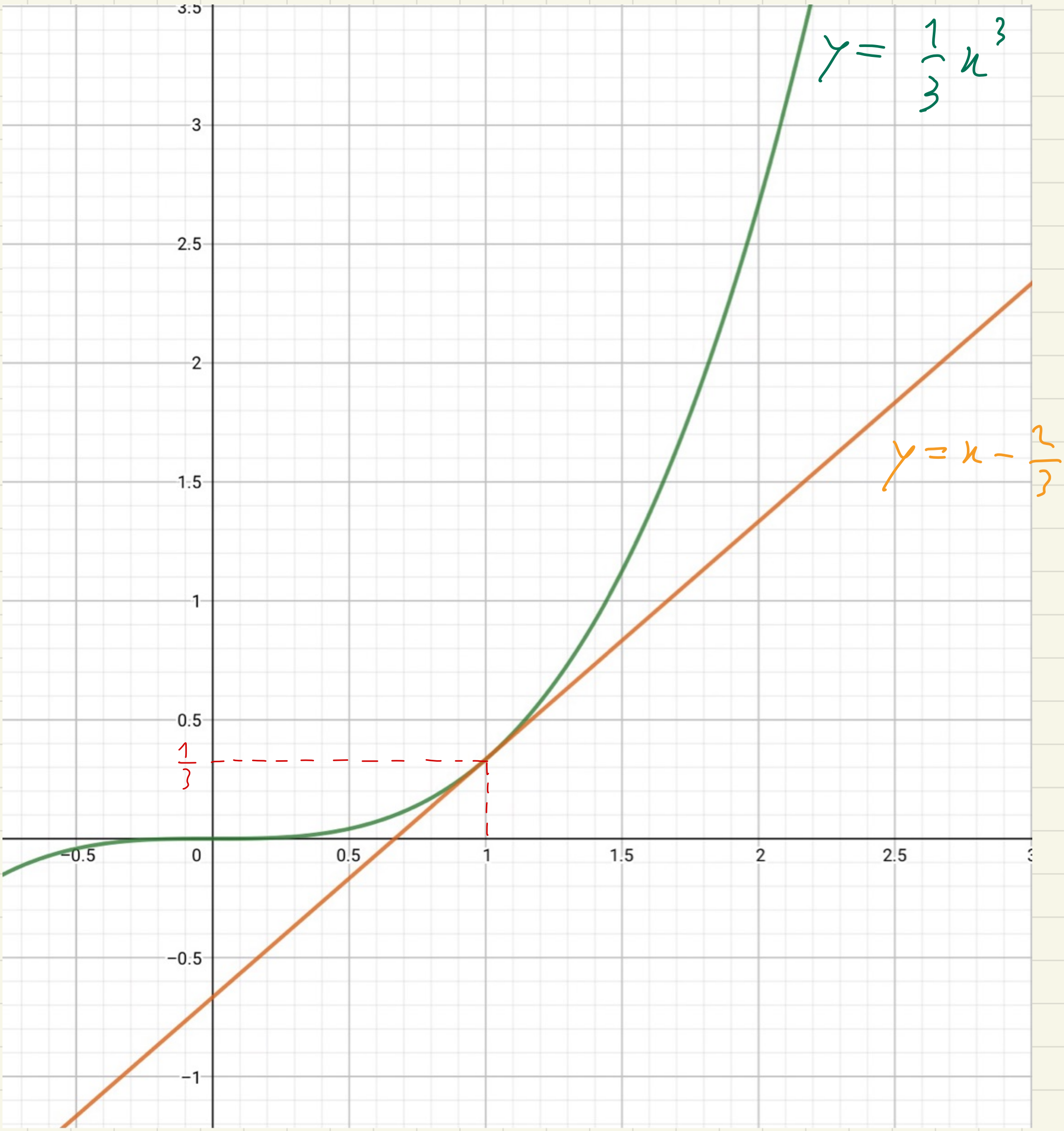
eq. di r:

$$y = f'(1)(x-1) + f(1)$$

$$= 1 \cdot (x-1) + \frac{1}{3} =$$

$$= x - \frac{2}{3}$$

$$y = x - \frac{2}{3}$$



y = 1/3 x^3
y = x - 2/3
1/3
3.5
3
2.5
2
1.5
1
0.5
0
−0.5
−1
0.5
1
1.5
2
2.5
3

TEOREMA (Algebra delle derivate):

$f, g : I \longrightarrow \mathbb{R}$

$x_0 \in I$

f, g derivabili in x_0.

Allora:

1. $f \pm g$ è derivabile in x_0 e
$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$

2. $f \cdot g$ è derivabile in x_0 e
$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

3. $\boxed{\frac{f}{g}}$ è derivabile in x_0 se $g(x_0) \neq 0$
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{(g(x_0))^2}$$

Esempio:

1

$$f(x) = x^5 + x^4 - 3x^2$$

$$Df = D\left(x^5 + x^4 - 3x^2\right) =$$

$$= D(x^5) + D(x^4) + D(-3x^2)$$

$$= 5x^4 + 4x^3 - 6x$$

2

$$D\left(\frac{x^2 - x + 1}{x^2+1}\right) =$$

$$= \frac{D(x^2 - x + 1)\cdot(x^2+1) - (x^2 - x + 1)\cdot D(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(2x-1)(x^2+1) - (x^2 - x + 1)\cdot 2x}{(x^2+1)^2}$$

③ Se f è derivabile e $f \neq 0$:

$$D\left(\frac{1}{f(x)}\right) = \frac{D(1)\cdot f(x) - 1\cdot Df}{(f(x))^2} =$$

$$= -\frac{Df(x)}{(f(x))^2}$$

Quindi:

$$D\left(\frac{1}{x^4+x^2+1}\right) = -\frac{D(x^4+x^2+1)}{(x^4+x^2+1)^2} =$$

$$= -\frac{4x^3+2x}{(x^4+x^2+1)^2}$$

$$D\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

• $D\, x^m = m\, x^{m-1} \quad (m \in \mathbb{N})$

$\Rightarrow D\, x^{-n} = D\left(\frac{1}{x^n}\right) = \qquad (x \neq 0)$

$= -\frac{D(x^n)}{(x^n)^2} =$

$= -\frac{n \cdot x^{n-1}}{x^{2n}} = -\frac{n}{x^{n+1}} =$

$= -n\, x^{-n-1}$

Derivabilità di alcune funzioni elementari:

<u>PROP.</u> 1 : $\sin x$, $\cos x$, $\operatorname{tg} x$ sono funzioni derivabili nel loro dominio naturale:

$$D \sin x = \cos x$$

$$D \cos x = -\sin x$$

$$D \operatorname{tg} x = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$$

<u>PROP.</u> 2 : La funzione a^x è derivabile in $\mathbb{R}$: $(0 < a,\ a \neq 1)$

$$D a^x = (\ln a) \cdot a^x \quad , \quad \left(D e^x = e^x \right)$$

Proveremo solo prop. 1 ,
mostrando che è una conseguenza
del limite fondamentale:

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

La prop. 2 è conseguenza del
limite (che non abbiamo dimostrato):

$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

DIM. (Prop. 1):

Mostriamo che la funzione seno è derivabile in x:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} =$$

(dalle formule di addizione)

$$= \lim_{h\to 0} \boxed{\frac{\sin x\cdot\cos h+\cos x\cdot\sin h-\sin x}{h}} =$$

$$= \frac{\sin x\cdot(\cos h-1)+\cos x\cdot\sin h}{h}$$

$$\frac{\sin x \cdot (\cos h - 1) + \cos x \cdot \sin h}{h} =$$

$$= \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}$$

$$= \sin x \cdot \frac{\cos h - 1}{h^2} \cdot h + \cos x \cdot \frac{\sin h}{h}$$

$$\frac{\cos h - 1}{h^2} \xrightarrow{h \to 0} \frac{1}{2} \qquad h \xrightarrow{h \to 0} 0 \qquad \frac{\sin h}{h} \xrightarrow{h \to 0} 1$$

(vedi lez 25/10)

$$\lim_{h \to 0} \boxed{\frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h}} = \cos x$$

$$\Longrightarrow \quad D \sin x = \cos x$$

<u>Esercizio</u>:

Provare in modo analogo che $D \cos x = -\sin x$.

Ne segue che $\operatorname{tg} x = \dfrac{\sin x}{\cos x}$ è derivabile, essendo il rapporto di funzioni derivabili.

$$D \operatorname{tg} x = D \frac{\sin x}{\cos x} =$$

$$= \frac{D \sin x \cdot \cos x - \sin x \cdot D \cos x}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= 1 + \left(\frac{\sin x}{\cos x}\right)^2$$

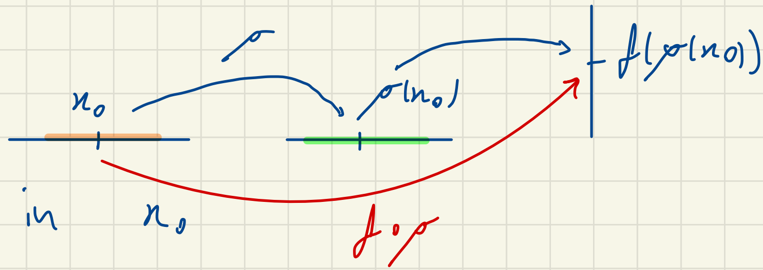
$$= 1 + (\operatorname{tg} x)^2$$

TEOREMA (Composizione di funzioni):

$I, J \subseteq \mathbb{R}$ intervalli

$f : I \longrightarrow \mathbb{R}$, $g : J \longrightarrow I$

$x_0 \in J$



g derivabile in x_0

f derivabile in $g(x_0)$

Allora $f \circ g$ è derivabile in x_0

e vale:

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$

Esempi:

$$g(x) = x^4 - 2x^2$$

$$f(y) = \sin y$$

1

$$h(x) = \sin\left(x^4 - 2x^2\right)$$

$$Dh(x) = (D\sin)\left(x^4 - 2x^2\right) \cdot D\left(x^4 - 2x^2\right) =$$

$$= \cos\left(x^4 - 2x^2\right) \cdot \left(4x^3 - 4x\right)$$

2

$$D\left(\frac{\cos x}{x+1}\right) = \frac{(D\cos x)\cdot(x+1) - \cos x \cdot D(x+1)}{(x+1)^2}$$

$$= \frac{-(\sin x)\cdot(x+1) - \cos x}{(x+1)^2}$$

(3)

$$D \operatorname{tg}(x^2+1) =$$

$$= (D \operatorname{tg})(x^2+1) \cdot D(x^2+1) =$$

$$= \frac{1}{\cos^2(x^2+1)} \cdot 2x = \frac{2x}{\cos^2(x^2+1)}$$

$f(y) = e^y \qquad g(x) = \cos x$

(4)

$$D\, e^{\cos x} = (D \exp)(\cos x) \cdot D \cos x$$

$$= e^{\cos x} \cdot (-\sin x)$$

(Attenzione:

$$D\, e^{f(x)} \neq e^{f(x)}$$

$$D\, e^{f(x)} = e^{f(x)} \cdot f'(x)$$

)

TEOREMA:

I intervallo $\subseteq \mathbb{R}$, $x_0 \in I$

$f : I \longrightarrow \mathbb{R}$

Si ha che:

f derivabile in x_0 $\overset{\Longrightarrow}{\not\Longleftarrow}$ f è continua in x_0

DIM.:

$x_0 \in D(I)$

Si tratta di provare che:

$$\lim_{x \to x_0} f(x) = f(x_0)$$ o equivalentemente

$$\lim_{x \to x_0} \left(f(x) - f(x_0) \right) = 0$$

$$\lim_{x \to x_0} \left(f(x) - f(x_0) \right) =$$

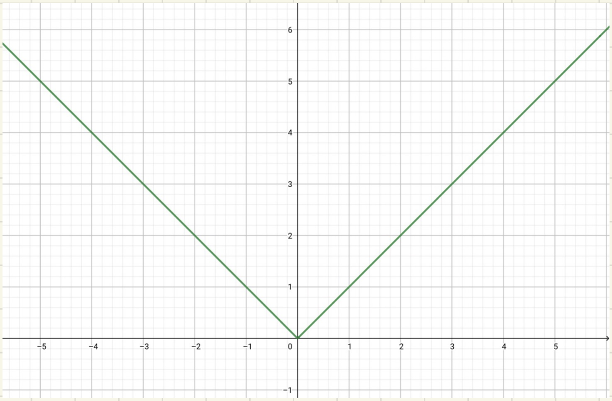
$$= \lim_{x \to x_0} \boxed{\frac{f(x) - f(x_0)}{x - x_0}} \cdot \boxed{(x - x_0)} = 0$$

$$\frac{f(x) - f(x_0)}{x - x_0} \xrightarrow{x \to x_0} f'(x_0) \qquad (x - x_0) \xrightarrow{x \to x_0} 0$$

<u>OSS.</u>: Non vale il viceversa:

$f(x) = |x|$ è continua (in $x = 0$)

ma non è derivabile in $x = 0$!



DERIVATE DI ORDINE SUPERIORE:

I intervallo

$f : I \longrightarrow \mathbb{R}$ derivabile su I

$\Longrightarrow \quad \exists f' : I \longrightarrow \mathbb{R}$

Se f' è derivabile in $x_0 \in I$:

$$f''(x_0) := \lim_{h \to 0} \frac{f'(x_0+h) - f'(x_0)}{h} \in \mathbb{R}$$

$$\left(= \frac{d^2 f}{dx^2}(x_0) = D^2 f(x_0) \right)$$

Derivata II di f in x_0

f si dice <u>derivabile 2 volte</u> in x_0

Se f è derivabile 2 volte $\forall x_0 \in I$

$\Longrightarrow \quad \exists f'': I \longrightarrow \mathbb{R}$

Se f'' è derivabile in $x_0 \in I$:

$$f'''(x_0) := \lim_{h \to 0} \frac{f''(x_0+h) - f''(x_0)}{h} \in \mathbb{R}$$

$$\parallel$$

$$\frac{d^3 f}{dx^3}(x_0) = D^3 f(x_0)$$

Derivata III di f in x_0

$\vdots$

$f', \quad f'', \quad f''', \quad f^{IV}, \quad f^{V}$

$\updownarrow \quad \updownarrow \quad \updownarrow$

$f^{(1)} \quad f^{(2)} \quad f^{(3)} \quad f^{(4)} \quad f^{(5)} \ldots \quad f^{(n)}$

Esempi:

$$f(x) = x^3 - x^4 + 2x + 1$$

$$f'(x) = 3x^2 - 4x^3 + 2$$

$$f''(x) = 6x - 12x^2$$

$$f'''(x) = 6 - 24x$$

$$f^{IV}(x) = -24$$

$$f^{V}(x) = 0$$

$$f^{(n)}(x) = 0 \qquad \text{se} \qquad n \geq 5$$

$$p(x) = \sum_{j=0}^{n} a_j x^j$$

$$p^{(k)}(x) \equiv 0 \qquad \text{se} \qquad k \geq n+1$$

$$f(x) = \sin x$$

$$D \sin x = \cos x$$

$$D^2 \sin x = D \cos x = -\sin x$$

$$D^3 \sin x = D(-\sin x) = -\cos x$$

$$D^4 \sin x = D(-\cos x) = \sin x$$

$$D^5 \sin x = \cos x$$

$$D^{n+4} \sin x = D^n \sin x \qquad \forall n$$

<u>Esercizio</u>:

$$D^{n+4} \cos x = D^n \cos x \qquad \forall n$$

$$D e^x = c^x$$

$$D^2 e^x = e^x$$

$$D^n e^x = e^x \qquad \forall n$$

$$D\left(a e^x\right) = a \cdot D e^x = a e^x$$

$$D^n\left(a e^x\right) = a c^x \qquad \forall n$$

NOTA:

La derivata f' viene spesso chiamata derivata prima.

DEF. (Classe C^k)

I intervallo

$$f \in C^k(I) \overset{def}{\iff} \begin{cases} f \text{ è derivabile } k\text{-volte su } I; \\ f^{(k)} \text{ è continua su } I \end{cases}$$

(Notaz.: $f^{(0)} \equiv f$)

Dunque:

$$C^0(I) = \{ f : I \to \mathbb{R} \mid f \text{ continua su } I \}$$

$$C^1(I) = \{ f : I \longrightarrow \mathbb{R} \mid f \text{ è derivabile } f' : I \to \mathbb{R} \text{ è continua} \}$$

$$C^2(I) = \{ f : I \longrightarrow \mathbb{R} \mid f \text{ è derivabile 2 volte } f'' : I \longrightarrow \mathbb{R} \text{ è continua} \}$$

$$\vdots$$

$$C^\infty(I) := \bigcap_k C^k(I)$$

Def.:

$$f \in C^k(I) \Rightarrow \begin{cases} f^{(j)} : I \to \mathbb{R} \\ 0 \leq j \leq k \\ \text{sono funzioni} \\ \text{continue} \end{cases}$$

$$C^{\infty}(I) \subsetneq C^{k}(I) \subsetneq \dots \subsetneq C^{2}(I) \subsetneq$$

$$\subsetneq C^{1}(I) \subsetneq C^{0}(I)$$

<u>Esempio</u>:

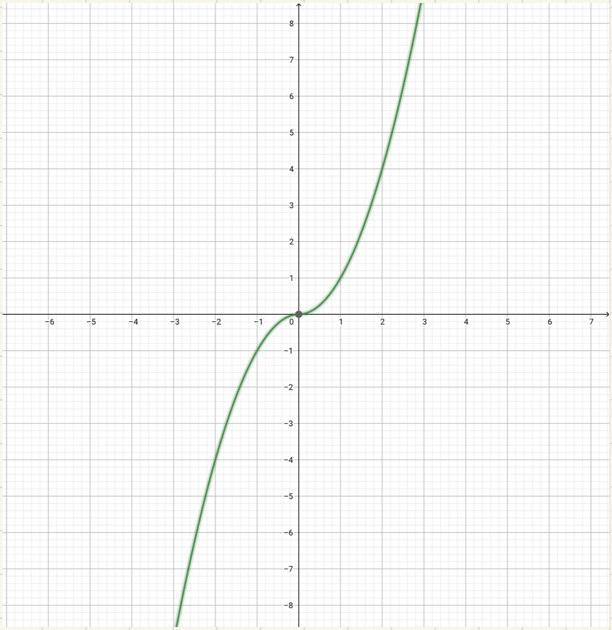
$$f(x) = |x| \longrightarrow f \in C^{0}(\mathbb{R})$$

$$f \notin C^{1}(\mathbb{R})$$

$$\varphi(x) = x|x| \longrightarrow \varphi \in C^{1}(\mathbb{R})$$

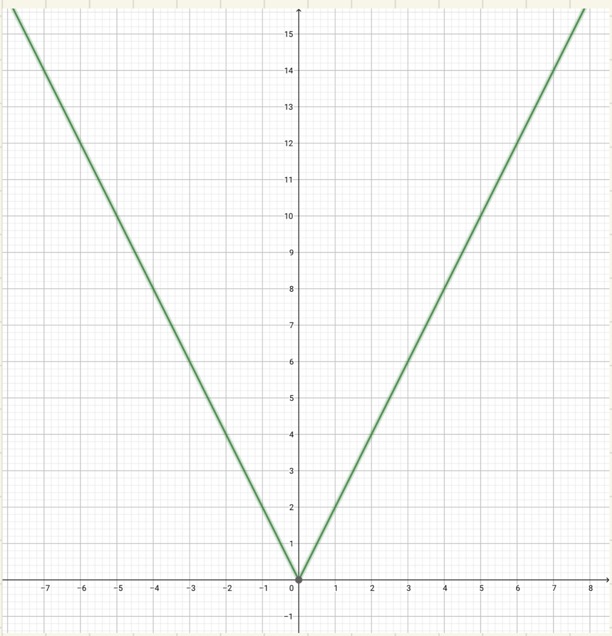
$$\varphi \notin C^{2}(\mathbb{R})$$

$$\left(h(x) = x^{k}|x| \Longrightarrow \begin{array}{l} h \in C^{k}(\mathbb{R}) \\ h \notin C^{k+1}(\mathbb{R}) \end{array} \right)$$



$$f(x) = x|x|$$

$$= \begin{cases} x^2 & \text{se } x \geq 0 \\ -x^2 & \text{se } x < 0 \end{cases}$$



$$f'(x) = \begin{cases} 2x & \text{se } x \geq 0 \\ -2x & \text{se } x < 0 \end{cases}$$

é continua

$$\underbrace{f''_-(0) = -2 \qquad f''_+(0) = 2}_{\neq}$$

TEOREMA (derivata della funzione inversa)

$f : \,]a, b[\, \longrightarrow \mathbb{R}$ continua, invertibile

$x_0 \in \,]a, b[$ $\qquad x_0 \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y_0 = f(x_0)$

f derivabile in x_0 : $f'(x_0) \neq 0$

$\Longrightarrow$ $f^{(-1)}$ è derivabile in $y_0 = f(x_0)$

e vale:

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(x_0)\Big|_{x_0 = f^{-1}(y_0)}}}$$

(Senza dimostrazione)

La parte "difficile" del teorema consiste nel mostrare che f^{-1} è derivabile in $y_0 = f(x_0)$ -
La formula $Df^{-1}(y_0)$ si ottiene immediatamente:

$$\begin{tikzcd} x_0 \arrow[r, bend left, "f"] & y_0 \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$$f^{-1}(f(x)) = x \qquad \forall x$$

Derivando ambo i membri:

$$D\left[f^{-1}(f(x))\right] = 1$$

$$D\left[f^{-1}(f(x))\right] = 1$$

$$\Longrightarrow$$

$$(Df^{-1})(f(x_0)) \cdot \overset{\neq 0}{\boxed{(Df)(x_0)}} = 1$$

$$\Longrightarrow$$

$$(Df^{-1})(f(x_0)) = \frac{1}{(Df)(x_0)}$$

$$y_0 = f(x_0) \longrightarrow x_0 = f^{-1}(y_0)$$

$$(Df^{-1})(y_0) = \frac{1}{Df(x_0)\Big|_{x_0 = f^{-1}(y_0)}}$$

Alcune applicazioni del teorema:

(1)

$$f : \mathbb{R} \longrightarrow \mathbb{R}_+$$

$$f(x) = a^x \qquad (0 < a, \; a \neq 1)$$

f è continua e invertibile

$$Df(x) = (\ln a)\, a^x \neq 0 \qquad \forall x \in \mathbb{R}$$

Dal teorema precedente:

(1) $f^{-1}(y) = \log_a y$ è derivabile $\forall y \in \mathbb{R}: y > 0$

(2) $$D \log_a y = D f^{(-1)}(y) = \frac{1}{(Df)(x)\Big|_{x = f^{-1}(y) = \log_a y}} =$$

$$= \frac{1}{(\ln a)\, a^x \Big|_{x = \log_a y}} =$$

$$D \log_a y = D f^{(-1)}(y) = \frac{1}{(Df)(x)\Big|_{x = f^{-1}(y) = \log_a y}} =$$

$$= \frac{1}{(\ln a)\, a^x \Big|_{x = \log_a y}} =$$

$$= \frac{1}{(\ln a) \cdot a^{\log_a y}} = \frac{1}{(\ln a) \cdot y}$$

$$\Longrightarrow$$

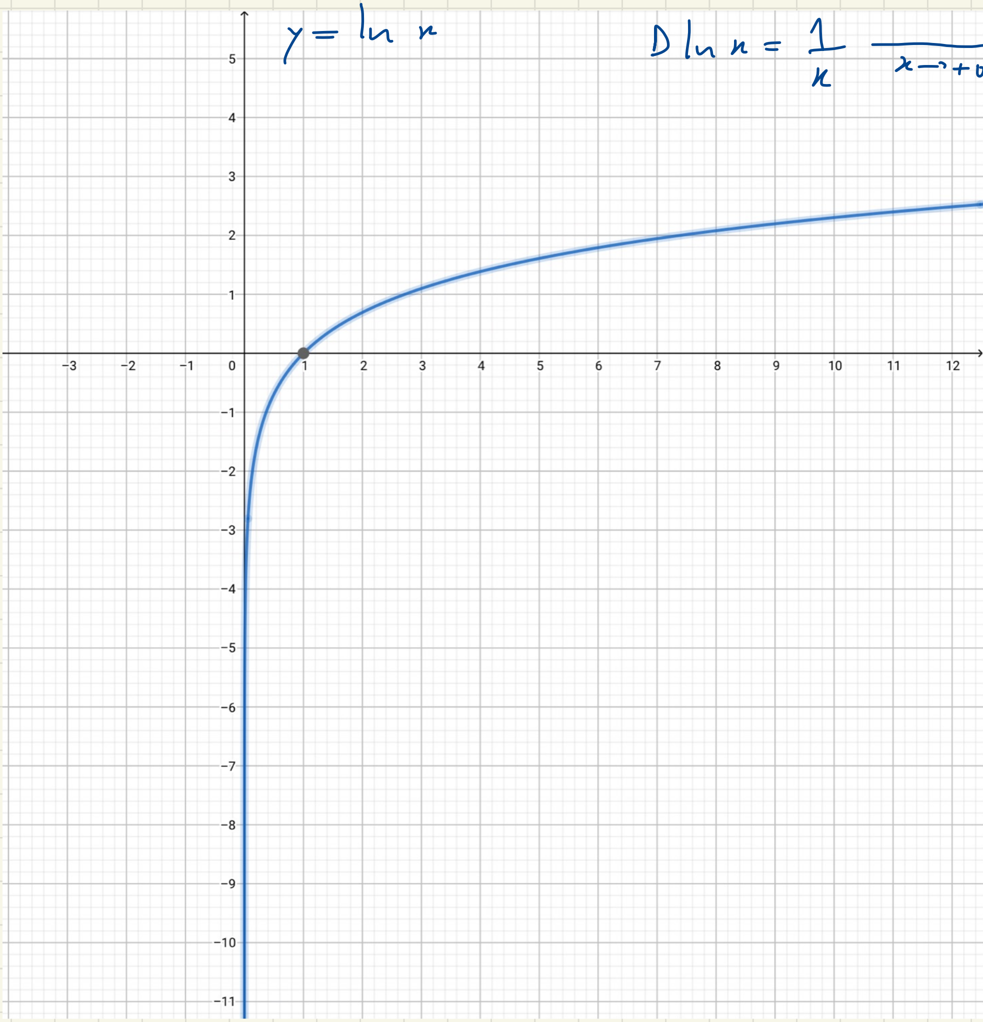
$$\boxed{D \log_a y = \frac{1}{\ln a} \cdot \frac{1}{y}}$$

$$D \log_a x = \frac{1}{\ln a} \cdot \frac{1}{x}$$

$$D \ln x = \frac{1}{x}$$

$y = \ln x$

$$D \ln x = \frac{1}{x} \xrightarrow[x \to +\infty]{} 0$$



$$\left[\ e^{\ln t} = t \quad \forall t > 0\ \right]$$

(2)

$$f(x) = x^{\alpha} \qquad (\alpha \in \mathbb{R}\ ,\quad x > 0)$$

$$= e^{\ln(x^{\alpha})} =$$

$$= e^{\alpha \cdot \ln x} \quad \bar{e} \text{ derivabile } \forall x > 0$$

$$D(x^{\alpha}) = D\left(e^{\alpha \ln x}\right) =$$

$$x^{\alpha} =$$

$$= e^{\alpha \ln x} \cdot D(\alpha \ln x) =$$

$$= x^{\alpha} \cdot \alpha \cdot \frac{1}{x}$$

$$= \alpha\, x^{\alpha - 1}$$

$$\boxed{D\, x^{\alpha} = \alpha\, x^{\alpha-1}}$$

$(x > 0)$

$(\alpha \in \mathbb{R})$

3

$$D \arcsin y = \,?$$

$f(x)$

$\shortparallel$

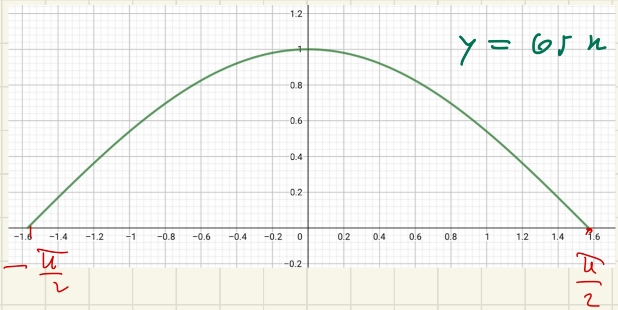
$$\sin\Big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \xrightarrow[\text{su}]{1-1} [-1, 1]$$

$$\arcsin := \left(\sin\Big|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}\right)^{-1} : [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$\shortparallel$

$f^{(-1)}$

$$D \sin x = \cos x \neq 0 \quad \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$



$$\sin\Big|_{\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[} : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow \left]-1, 1\right[$$

$$f^{-1}(y) = \left(\sin\Big|_{\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[}\right)^{-1} = \arcsin y$$

$\Longrightarrow$

- $\arcsin : \left]-1, 1\right[\longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$
- $\arcsin y$ è derivabile se $y \in \left]-1, 1\right[$

$$\forall y \in \,]-1, 1[\, : \left(\longrightarrow -\frac{\pi}{2} < \arcsin y < \frac{\pi}{2} \right)$$

$$D \arcsin y = \frac{1}{(D \sin x)\Big|_{x = f^{-1}(y) = \arcsin y}} =$$

$$= \frac{1}{\cos x \Big|_{x = \arcsin y}} =$$

$$= \frac{1}{\cos(\arcsin y)}$$

$$\cos x = \pm\sqrt{1 - \sin^2 x}$$

Quale segno?

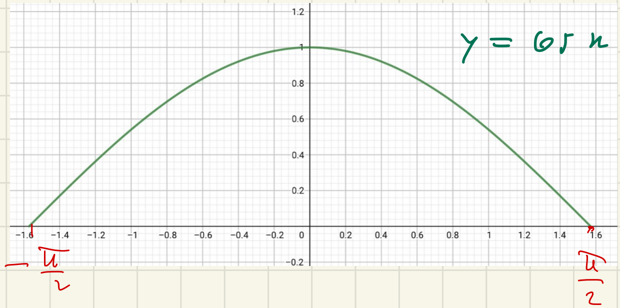
$$(\cos^2 x + \sin^2 x = 1)$$

$$D \arcsin y = \frac{1}{\cos(\arcsin y)} =$$

$$-\frac{\pi}{2} < \boxed{\arcsin y} < \frac{\pi}{2}$$

$$\Downarrow$$

$$\cos(\arcsin y)$$



$$\cos(\arcsin y) = \pm\sqrt{1-(\sin(\arcsin y))^2} =$$

$$= \sqrt{1-y^2}$$

$$\Longrightarrow$$

$$\boxed{D \arcsin y = \frac{1}{\sqrt{1-y^2}} \qquad (-1<y<1)}$$

(4) (Da svolgere come esercizio!)

$$D \arccos y = ?$$

$$f(x) = \cos\big|_{[0,\pi]} : [0,\pi] \longrightarrow [-1,1]$$

$$D\cos x = \sin x \neq 0 \quad \forall x \in]0,\pi[$$

$$f^{-1}(y) = \left(\cos\big|_{]0,\pi[}\right)^{-1} = \arccos y$$

$\arccos y$ è derivabile se $y \in]-1,1[$

$$\forall y \in \left]-1, 1\right[\; : \quad \left(\longrightarrow \; -\frac{\pi}{2} < \arccos y < \frac{\pi}{2} \right)$$

$$D \arccos y = \frac{1}{\left.(D \cos x)\right|_{x = f^{-1}(y)}} =$$

$$f^{-1}(y) = \arccos y$$

$$= \frac{1}{\left.-\sin x\right|_{x = \arccos y}} =$$

$$= \frac{1}{-\sin(\arccos y)}$$

$$\sin x = \pm\sqrt{1 - \cos^2 x}$$

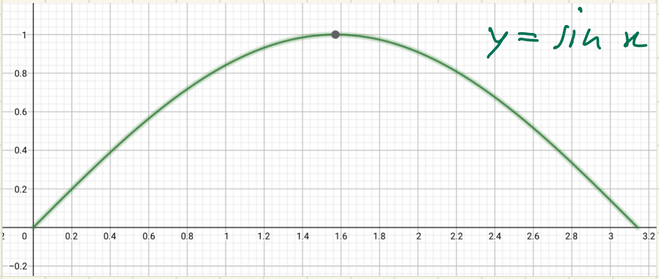
Quale segno!

$$D \arccos y = \frac{1}{-\sin(\arccos y)} =$$

$$0 < \boxed{\arccos y} < \pi$$

$$\Downarrow$$

$$\sin(\arccos y)$$



$$\sin(\arccos y) = \pm\sqrt{1-(\cos(\arccos y))^2} =$$

$$= \sqrt{1-y^2} \qquad (-1<y<1)$$

$$\Longrightarrow$$

$$\boxed{D \arccos y = -\frac{1}{\sqrt{1-y^2}} \qquad (-1<y<1)}$$

(5)

$$D \operatorname{arctg} y = ?$$

$$\underbrace{\operatorname{tg}\big|_{]-\frac{\pi}{2}, \frac{\pi}{2}[}}_{f} : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\xrightarrow[\text{su}]{1-1} \mathbb{R}$$

$$f^{-1}(y) = \operatorname{arctg} y$$

$$D \operatorname{tg} x = 1 + \operatorname{tg}^2 x \neq 0 \quad \forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

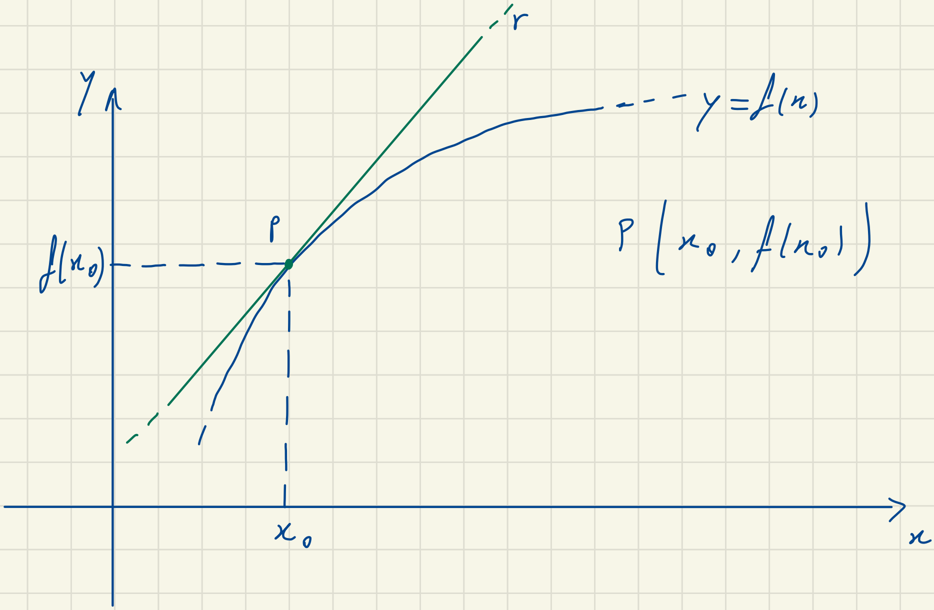
$$D \operatorname{arctg} y = \frac{1}{D \operatorname{tg} x \big|_{x = f^{-1}(y) = \operatorname{arctg} y}} =$$

$$= \frac{1}{1 + \operatorname{tg}^2 x \big|_{x = \operatorname{arctg} y}} =$$

$$= \frac{1}{1 + (\operatorname{tg}(\operatorname{arctg} y))^2} = \frac{1}{1 + y^2}$$

$$\boxed{D \arctan x = \frac{1}{1+x^2}, \quad x \in \mathbb{R}}$$

$$r: \quad y = f'(x_0)(x - x_0) + f(x_0)$$



Esercizio:

Calcolare la retta tangente al grafico di $f(x) = \ln(\cos x + \sin x)$ in $x = \frac{\pi}{2}$.

$$f\left(\frac{\pi}{2}\right) = \ln\left(\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) =$$

$$= \ln 1 = 0$$

$$D \ln(\cos x + \sin x) = (D \ln)(\cos x + \sin x) \cdot$$
$$\cdot D(\cos x + \sin x)$$

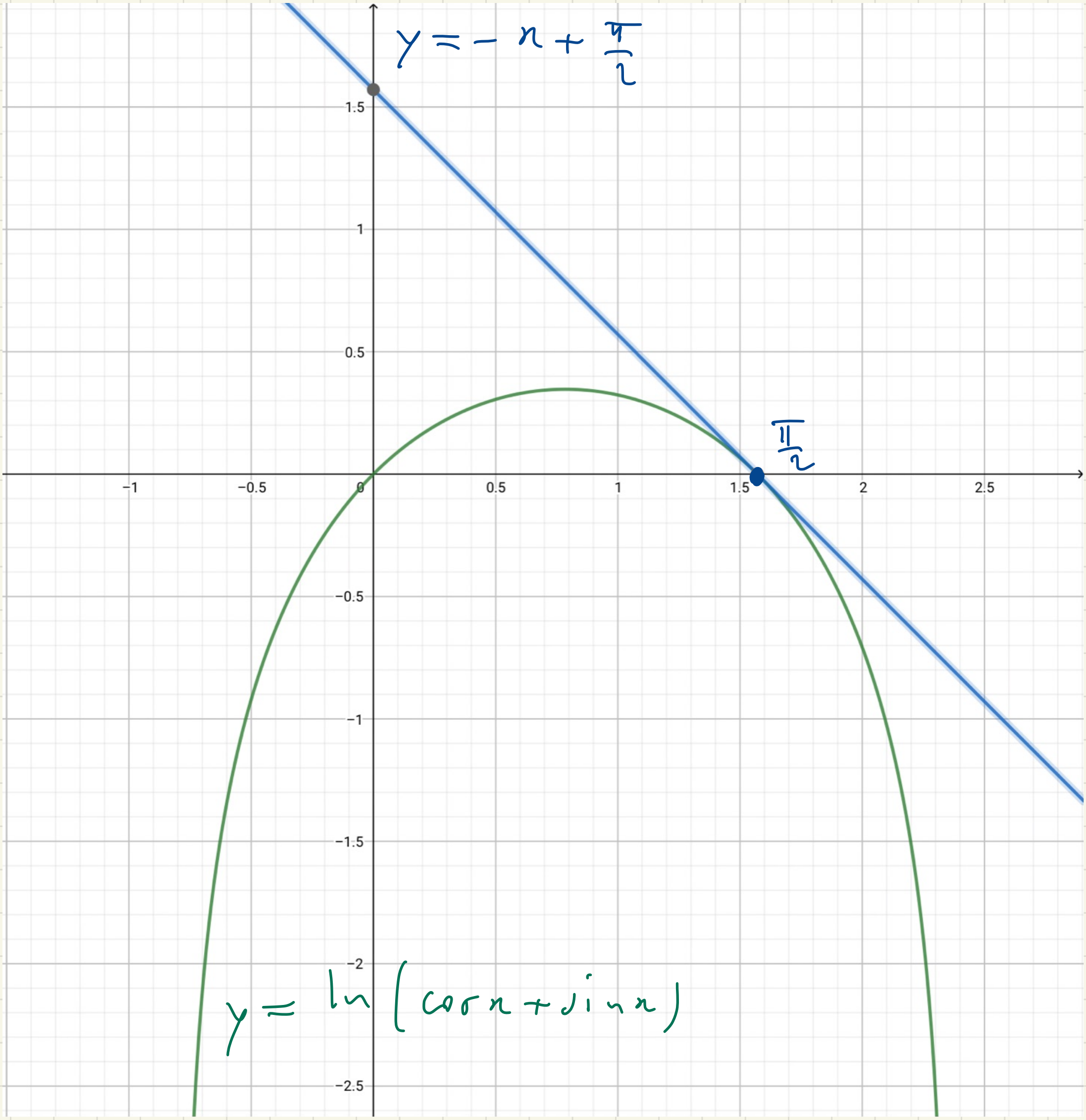
$$f'(x) = \frac{1}{\cos x + \sin x} \cdot (-\sin x + \cos x)$$

$$f'\left(\frac{\pi}{2}\right) = \frac{1}{0+1} \cdot (-1+0) = -1$$

rett. tangente in $x = \frac{\pi}{2}$:

$$y = f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right)$$

$$y = -\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$$


$y = -x + \frac{\pi}{2}$
$\frac{\pi}{2}$
$y = \ln(\cos x + \sin x)$
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0.5
1
1.5
2
2.5

Esercizi (I)

① Calcolare la retta tangente alla curva:

ⓐ $y = x^2 - 3x + 2$ in $x = 3$

$[\ 3x - y - 7 = 0\ ]$

ⓑ $yx = 4$ in $x = 1$

$[\ 4x + y - 8 = 0\ ]$

ⓒ $y = 3\cos^2 x - 2\sin x + 5x$ in $x = 0$

$[\ y = 3x + 3\ ]$

ⓓ $y = \arctan(2x)$ in $x = -\dfrac{1}{2\sqrt{3}}$

$[\ 18x - 12y = 2\pi - 3\sqrt{3}\ ]$

(2) Determinare le equazioni delle rette tangenti a $y = x^2 - 3x + 2$ nei punti in cui la parabola incontra gli assi:

$[\,y = -x + 1, \quad y = x - 2, \quad y = -3x + 2\,]$

(3) Determinare $K \in \mathbb{R}$, in modo che la tangente in $x = 2$ alla parabola $y = 2x^2 - (4K+1)x + 2K$ sia parallela alla retta $y = 3x - 6$

$[K = 1]$

Alcuni esempi sul calcolo della derivata:

(A)

$$D\left(2\sqrt{x}+3\sqrt[3]{x}\right)=D\left(2x^{\frac{1}{2}}+3x^{\frac{1}{3}}\right)$$

$$=2\cdot\frac{1}{2}\cdot x^{\frac{1}{2}-1}+3\cdot\frac{1}{3}\cdot x^{\frac{1}{3}-1}$$

$$=x^{-\frac{1}{2}}+x^{-\frac{2}{3}}=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x^2}}=$$

$$=\frac{\sqrt{x}+\sqrt[3]{x}}{x}$$

(B)

$$D\,\frac{x^2-4}{x^2+4}=\frac{D(x^2-4)\cdot(x^2+4)-D(x^2+4)\,(x^2-4)}{(x^2+4)^2}$$

$$=\frac{2x(\cancel{x^2}+4)-2x(\cancel{x^2}-4)}{(x^2+4)^2}=\frac{16x}{(x^2+4)^2}$$

C $D\left(x^3 \cdot \ln x\right) =$

$$= D(x^3) \cdot \ln x + x^3 \, D \ln x =$$

$$= 3x^2 \ln x + x^2$$

D $D \cos^2 x = 2 \cos x \cdot D \cos x =$

$$= 2 \cos x \cdot (-\sin x)$$

$$= -2 \sin x \cdot \cos x = -\sin(2x)$$

E $D\left(\sin x \cdot \operatorname{tg} x + \cos x\right) =$

$$= D \sin x \cdot \operatorname{tg} x + \sin x \cdot D \operatorname{tg} x + (-\sin x)$$

$$= \underbrace{\cos x \cdot \operatorname{tg} x}_{u} + \sin x \cdot \frac{1}{\cos^2 x} - \sin x$$

$$\cancel{\cos} x \cdot \frac{\sin x}{\cancel{\cos x}}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\operatorname{tg} x}{\cos x}$$

F

$$D\left(\frac{\ln x}{x^n}\right) = \frac{D\ln x \cdot x^n - Dx^n \cdot \ln x}{x^{2n}} =$$

$$= \frac{x^{n-1} - n \cdot x^{n-1} \cdot \ln x}{x^{2n}} =$$

$$= \frac{\cancel{x^{n-1}}\,(1 - n \ln x)}{x^{2n-(n-1)}} \qquad x^{2n-n+1} = x^{n+1}$$

$$= \frac{1 - n \cdot \ln x}{x^{n+1}}$$

G

$$D\,\frac{1}{\ln x} = -\frac{1}{(\ln x)^2} \cdot \frac{1}{x} = -\frac{1}{x \cdot (\ln x)^2}$$

H

$$D\,\frac{1 - \ln x}{1 + \ln x} \qquad \left[\frac{-2}{x\,(1 + \ln x)^2}\right]$$

(H)

$$D\,\frac{1-\ln x}{1+\ln x} =$$

$$= \frac{D(1-\ln x)\cdot(1+\ln x) - D(1+\ln x)\cdot(1-\ln x)}{(1+\ln x)^2}$$

$$= \frac{-\frac{1}{x}\cdot(1+\ln x) - \frac{1}{x}(1-\ln x)}{(1+\ln x)^2}$$

$$= \frac{-\frac{1}{x} \cancel{- \frac{1}{x}\ln x} - \frac{1}{x} \cancel{+ \frac{1}{x}\ln x}}{(1+\ln x)^2} =$$

$$= \frac{-\frac{2}{x}}{(1+\ln x)^2} = -\frac{2}{x(1+\ln x)^2}$$

I $\quad D\, e^{-x} = -e^{-x}$

L $\quad D\, 3^{x} = 3^{x} \ln 3$

M $\quad D\, \dfrac{e^{x}}{x^{2}} \qquad \left[e^{x} \dfrac{x-2}{x^{3}} \right]$

N $\quad D\, \dfrac{1+e^{x}}{1-e^{x}} = \dfrac{e^{x}(1-e^{x}) - (-e^{x})(1+e^{x})}{(1-e^{x})^{2}} =$

$$= \frac{2e^{x}}{(1-e^{x})^{2}}$$

O $\quad D\, \sin\left(\dfrac{1}{x}\right) = \cos\left(\dfrac{1}{x}\right) \cdot -\dfrac{1}{x^{2}} =$

$$= -\frac{\cos\left(\frac{1}{x}\right)}{x^{2}}$$

M

$$D \frac{e^x}{x^2} = \frac{De^x \cdot x^2 - Dx^2 \cdot e^x}{(x^2)^2} =$$

$$= \frac{e^x \cdot x^2 - 2x \cdot e^x}{x^4} =$$

$$= \frac{\cancel{x}\, e^x (x-2)}{x^{\cancel{4}\,3}} =$$

$$= \frac{e^x (x-2)}{x^3}$$

$$D \left(\frac{[e^x (x-2)]}{x^3} \right) =$$

$$= \frac{D[e^x (x-2)] \cdot x^3 - Dx^3 \cdot e^x (x-2)}{x^6} =$$

$$= \frac{(De^x \cdot (x-2) + e^x D(x-2)) \cdot x^3 - 3x^2 e^x (x-2)}{x^6}$$

$$= \frac{\left(D e^{x} \cdot (x-2) + e^{x} D(x-2)\right) \cdot x^{3} - 3x^{2} e^{x} (x-2)}{x^{6}}$$

$$= \frac{\left(e^{x} \cdot (x-2) + e^{x}\right) x^{3} - 3x^{2} e^{x} (x-2)}{x^{6}}$$

$$= \frac{x^{3} e^{x} (x-2) + x^{3} e^{x} - 3x^{2} e^{x} (x-2)}{x^{6}}$$

$$= \frac{e^{x} (x^{3} - 3x^{2})(x-2) + x^{3} e^{x}}{x^{6}}$$

$$= \frac{e^{x} \left[(x^{3} - 3x^{2})(x-2) + x^{3} \right]}{x^{6}}$$

$$= \frac{e^{x} \left(x^{4} - \cancel{2x^{3}} - \cancel{3x^{3}} + 6x^{2} \cancel{+ x^{3}} \right)}{x^{6}}$$

$$= \frac{e^{x} (x^{4} + 6x^{2})}{x^{6}} = \frac{e^{x} (x^{2} + 6)}{x^{4}}$$

(P)

$$D\left(\sin^2 x \cdot \sin(x^2)\right) =$$

$$= 2\sin x \cdot \cos x \cdot \sin(x^2) +$$

$$+ \sin^2 x \cdot \cos(x^2) \cdot 2x =$$

$$= 2\sin x\left(\cos x \cdot \sin(x^2) + x\sin x \cdot \cos(x^2)\right)$$

(Q)

$$D\left(1+\sin^2 x\right)^4 = 4\left(1+\sin^2 x\right)^3 \cdot$$

$$\cdot\, 2\sin x \cdot \cos x =$$

$$= 4\left(1+\sin^2 x\right)^3 \sin(2x)$$

(R)

$$D\cos(x^3-1)^7 = -\sin(x^3-1)^7 \cdot 7(x^3-1)^6 \cdot 3x^2$$

$$D \sin^2 x = D(\sin x \cdot \sin x) =$$

$$= D\sin x \cdot \sin x + \sin x \cdot D\sin x$$

$$= \cos x \cdot \sin x + \sin x \cdot \cos x$$

$$= 2 \sin x \cdot \cos x$$

$$\sin^2 x := (\sin x)^2$$

$$\sin x^2 := \sin(x^2)$$

(S) $D \sin(\cos x) =$

$$= \cos(\cos x) \cdot (-\sin x)$$

(T) $D \ln(\ln x) = \dfrac{1}{\ln x} \cdot \dfrac{1}{x} =$

$$= \frac{1}{x \ln x}$$

(U) $D\, e^{\sin x} = e^{\sin x} \cdot \cos x$

(V) $D \ln \dfrac{e^x}{1+e^x} = \dfrac{1}{\dfrac{e^x}{1+e^x}} \cdot \dfrac{e^x(1+e^x) - e^{2x}}{(1+e^x)^2}$

$$= \frac{1+e^x}{e^x} \cdot \frac{e^x}{(1+e^x)^2} = \frac{1}{1+e^x}$$

(t)

$$D\, 2^{\frac{x}{\ln x}} = 2^{\frac{x}{\ln x}} \cdot \ln 2 \cdot D\left(\frac{x}{\ln x}\right)$$

$$\frac{\ln x - \frac{1}{x} \cdot x}{\ln^2 x}$$

(J)

$$D\, f(x)^{g(x)} = D\left(e^{\ln\left(f(x)^{g(x)}\right)} \right) =$$

$$= D\left(e^{g(x) \ln f(x)}\right) =$$

$$= e^{g(x) \ln f(x)} \cdot D\left(g(x)\left(\ln f(x)\right)\right)$$

$$= f(x)^{g(x)} \cdot D\left(g(x)\left(\ln f(x)\right)\right)$$

$$D\left(e^{x^2+1}\right) = (D\exp)(x^2+1) \cdot D(x^2+1)$$

$$= e^{x^2+1} \cdot 2x$$

$$D\, x^x = D\left(e^{\ln x^x}\right) = D\left(e^{x \ln x}\right)$$

$$= e^{x \ln x} \cdot D(x \ln x) =$$

$$= x^x \cdot \left(\ln x + x \cdot \frac{1}{x}\right)$$

$$= x^x \cdot (\ln x + 1)$$

$$D\, x^{\sin x} = x^{\sin x} \cdot D(\sin x \cdot \ln x)$$

$$\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

ESERCIZI (II):

(4) Calcolare le derivate delle seguenti funzioni:

(a) $y = 3\sqrt[3]{x} - \frac{3}{2}\sqrt[3]{x^2} + 1$ $\left[\frac{1}{\sqrt[3]{x^2}} - \frac{1}{\sqrt[3]{x}} \right]$

(b) $y = \frac{3x^2+1}{x-1}$ $\left[\frac{3x^2-6x-1}{(x-1)^2} \right]$

(c) $y = \frac{2}{x^3-1}$ $\left[-\frac{6x^2}{(x^3-1)^2} \right]$

(d) $y = \frac{x-\sqrt{x}}{x+\sqrt{x}}$ $\left[\frac{\sqrt{x}}{(x+\sqrt{x})^2} \right]$

(e) $y = \left(2x + \frac{5}{x}\right)^3$ $\left[3\left(2x+\frac{5}{x}\right)^2\left(2-\frac{5}{x^2}\right) \right]$

$$D\ 3\sqrt[3]{x} - \frac{3}{2}\sqrt[3]{x^2} + 1 =$$

$$= D\left(3x^{\frac{1}{3}} - \frac{3}{2}\cdot x^{\frac{2}{3}} + 1\right) =$$

$$= 3\cdot\frac{1}{3}x^{\frac{1}{3}-1} - \frac{3}{2}\cdot\frac{2}{3}x^{\frac{2}{3}-1} =$$

$$= x^{-\frac{2}{3}} - x^{-\frac{1}{3}} =$$

$$= \frac{1}{\sqrt[3]{x^2}} - \frac{1}{\sqrt[3]{x}}$$

f) $y = \left(\dfrac{x+1}{x-1}\right)^2 \qquad \left[-\dfrac{4(x+1)}{(x-1)^3} \right]$

g) $y = tg\, x - x \qquad \left[tg^2 x \right]$

h) $y = \dfrac{tg\, x}{x} \qquad \left[\dfrac{x - \sin x \cdot \cos x}{x^2 \cdot \cos^2 x} \right]$

i) $y = 3 \sin^4 - 2 \sin^6 x \qquad \left[12 \sin^3 x \cos^3 x \right]$

l) $y = \ln^2 x - \ln(\ln x) \qquad \left[\dfrac{2 \ln x}{x} - \dfrac{1}{x \ln x} \right]$

m) $y = x^{(x^2)} \qquad \left[x^{x^2+1} (2 \ln x + 1) \right]$

n) $y = (\sin x)^{\cos x} \qquad \left[(\sin x)^{\cos x} \cdot \left(\dfrac{\cos^2 x}{\sin x} - \sin x \ln \sin x \right) \right]$

o) $y = \operatorname{arctg} x^2$ $\qquad \left[\dfrac{2x}{1+x^4} \right]$

p) $y = \sin(\arccos x)$ $\qquad \left[-\dfrac{x}{\sqrt{1-x^2}} \right]$

q) $y = \operatorname{arctg}\left(\dfrac{\sin x}{1+\cos x} \right)$ $\qquad \left[\dfrac{1}{2} \right]$

r) $y = \arcsin \dfrac{x}{\sqrt{1+x^2}} - \operatorname{arctg} x$ $\qquad [0]$

s) $y = x \arcsin x + \sqrt{1-x^2}$ $\qquad [\arcsin x]$

(9)

$$y = \operatorname{arctg}\left(\frac{\sin x}{1+\cos x}\right) \qquad \left[\ \frac{1}{2}\ \right]$$

$$D(\ ) = \frac{1}{1+\left(\frac{\sin x}{1+\cos x}\right)^2} \cdot$$

$$\cdot \frac{\cos x \cdot (1+\cos x) - (-\sin x)\cdot \sin x}{(1+\cos x)^2} =$$

$$= \frac{1}{\frac{(1+\cos x)^2+\sin^2 x}{\cancel{(1+\cos x)^2}}} \cdot \frac{\cos x + \overbrace{\cos^2 x + \sin^2 x}^{1}}{\cancel{(1+\cos x)^2}} =$$

$$= \frac{1+\cos x}{1+2\cos x + \underbrace{\boxed{\cos^2 x + \sin^2 x}}_{=1}} =$$

$$= \frac{1+\cos x}{2+2\cos x} = \frac{1}{2}$$

r

$$D\left(\arcsin\frac{x}{\sqrt{1+x^2}} - \operatorname{arctg} x\right) =$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} \cdot \frac{\sqrt{1+x^2} - \frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x \cdot x}{1+x^2} -$$

$$- \frac{1}{1+x^2}$$

$$= \frac{1}{\sqrt{\frac{1+\cancel{x^2}-\cancel{x^2}}{1+x^2}}} \cdot \frac{\frac{(1+\cancel{x^2}) - \cancel{x^2}}{\sqrt{1+x^2}}}{1+x^2} - \frac{1}{1+x^2} =$$

$$= \frac{1}{\frac{1}{\cancel{\sqrt{1+x^2}}}} \cdot \frac{1}{\cancel{\sqrt{1+x^2}} \cdot (1+x^2)} - \frac{1}{1+x^2} = 0$$

La prossima lez.

è il 13 Novembre 2020