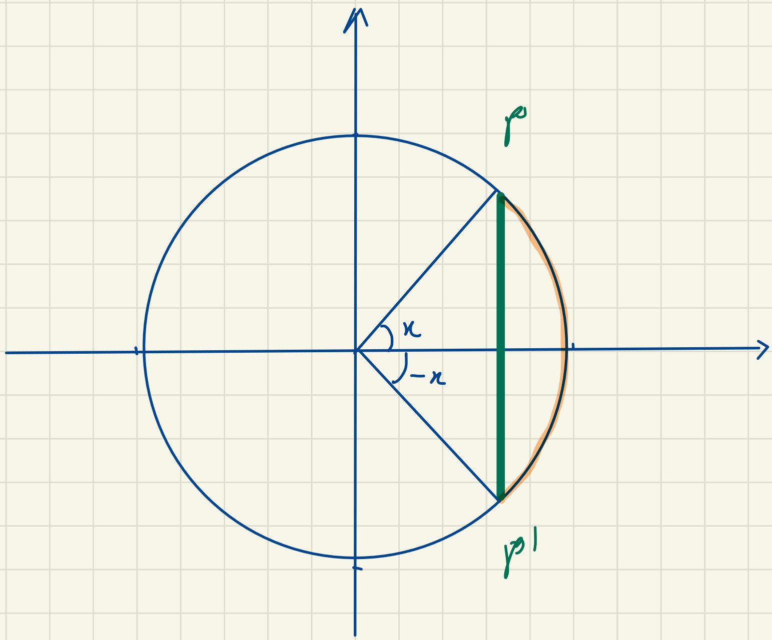
25. Ottobre. 2021

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

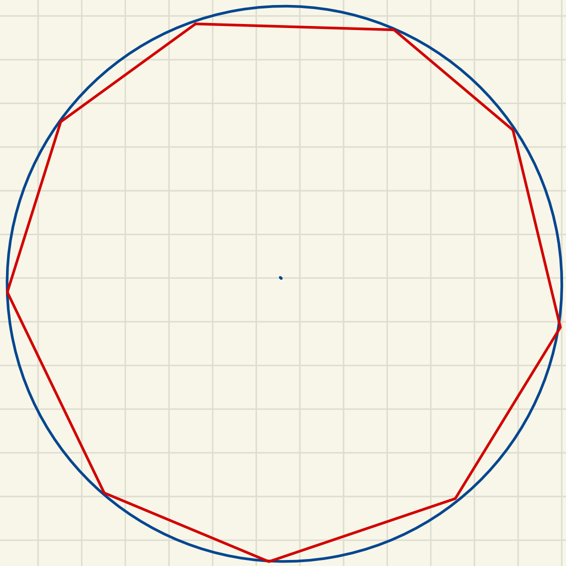
Interpretazione geometrica:



$\overline{PP'} = 2\sin x$ $\qquad$ $|\widehat{PP'}| = 2x$

$x \to 0$ $\qquad$ $\underline{2\sin x \sim 2x}$

PROBLEMA: Come si calcola l'area del cerchio?

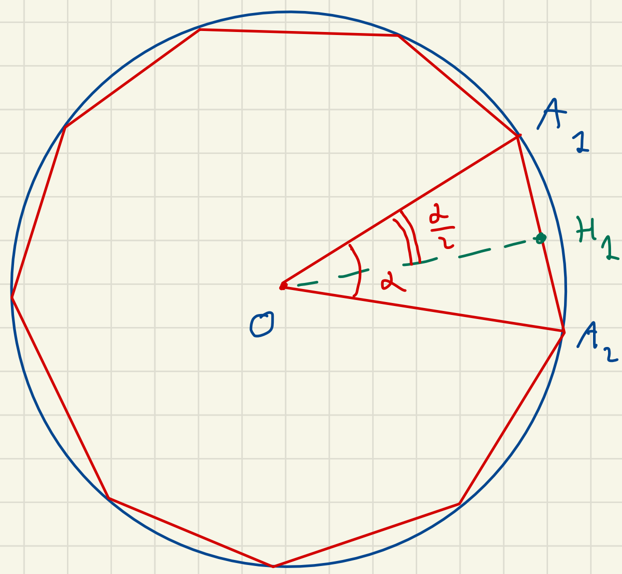


$\mathcal{C}$ cerchio di raggio r

$\mathcal{S}_n$ poligono regolare inscritto di n lati inscritto nel cerchio

$$\mathcal{A}(\mathcal{C}) = \lim_{n \to +\infty} \mathcal{A}(\mathcal{S}_n)$$

area



$$\alpha = \frac{2\pi}{n} \longrightarrow \frac{\alpha}{2} = \frac{\pi}{n}$$

$$\overline{OH_1} = \overline{OA_1} \cdot \cos\frac{\alpha}{2} = r \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\overline{A_1H_1} = \overline{OA_1} \cdot \sin\frac{\alpha}{2} = r \cdot \sin\left(\frac{\pi}{n}\right)$$

$$A\left(O\overset{\Delta}{A_1}A_2\right) = 2 \cdot A\left(O\overset{\Delta}{A_1}H_1\right) =$$

$$= \overline{A_1H_1} \cdot \overline{OH_1} =$$

$$= r^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$$

$$A(S_n) = n \cdot A(O\hat{A}_1 A_2) =$$

$$= n \cdot r^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\lim_{n \to +\infty} A(S_n) = ?$$

$$A(S_n) = r^2 \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \cdot \cos\left(\frac{\pi}{n}\right) =$$

$$= r^2 \cdot \pi \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \xrightarrow{n \to +\infty} 1 \qquad \cos\left(\frac{\pi}{n}\right) \xrightarrow{n \to +\infty} 1$$

$n \to +\infty$

$\Downarrow$

$\frac{\pi}{n} \to 0$

$$A(\mathcal{C}) = \lim_{n \to +\infty} A(S_n) = \pi \cdot r^2$$

Esercizio:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = ?$$

$$\frac{1-\cos x}{x^2} = \frac{1-\cos^2 x}{x^2} \cdot \frac{1}{1+\cos x}$$

$$= \left(\frac{\sin x}{x}\right)^2 \cdot \frac{1}{1+\cos x}$$

$$\downarrow \qquad\qquad \downarrow$$

$$1 \qquad\qquad \frac{1}{1+1} = \frac{1}{2}$$

Quindi:

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$$

Un secondo limite notevole è:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

(senza dimostrazione)

$(0 < a, \quad a \neq 1)$

(caso particolare: $a = e$)

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \ln e = 1$$

LIMITE INFINITO

AL FINITO:

Esempio:

$$f(x) = \frac{1}{x^2 - 6x + 9}$$

$$D(f) = \mathbb{R} \setminus \{3\}$$

3 è di accumulazione

per $\mathbb{R} \setminus \{3\}$

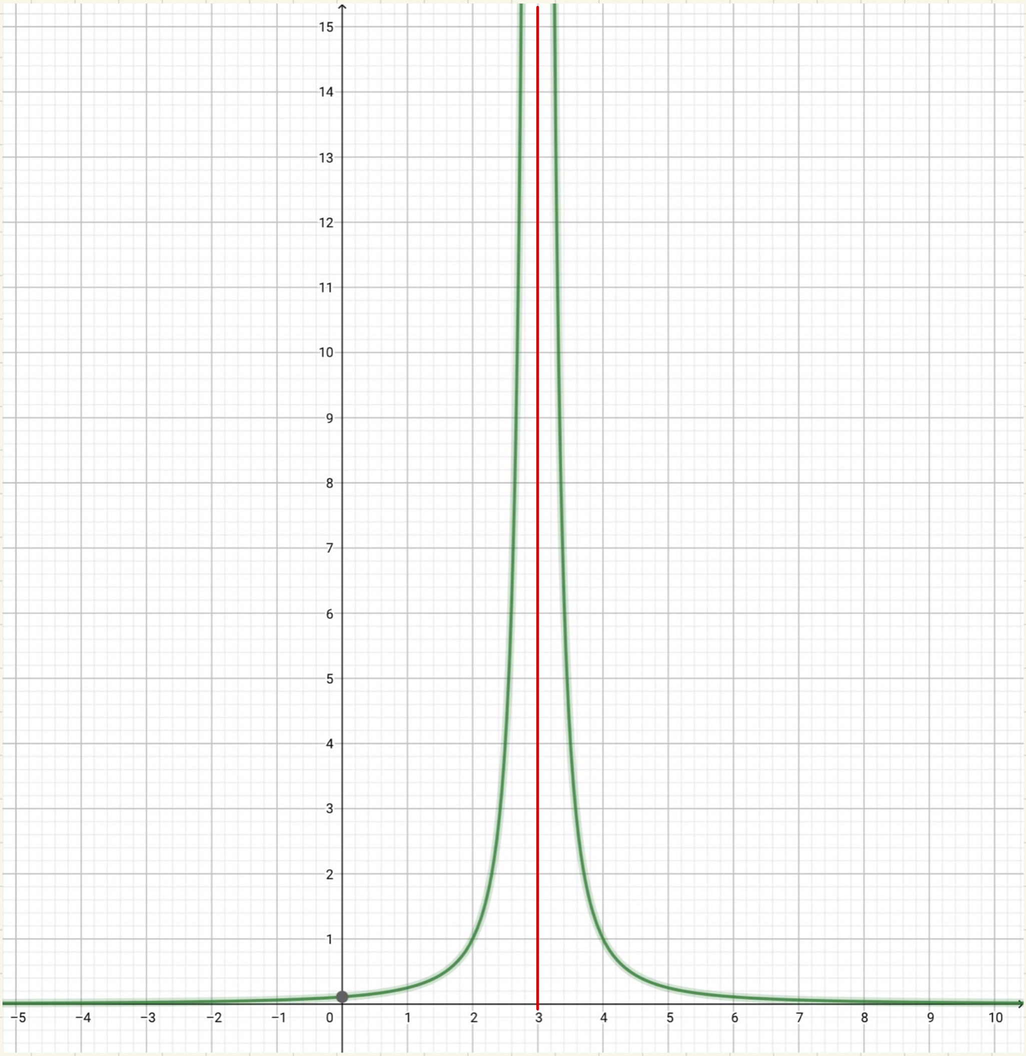
x	$\frac{1}{x^2-6x+9}$
$3+\frac{1}{10}$	100
$3+\frac{1}{100}$	$10'000$
$3+\frac{1}{1'000}$	$1'000'000$
$\vdots$	$\vdots$

x	$\frac{1}{x^2-6x+9}$
$3-\frac{1}{10}$	100
$3-\frac{1}{100}$	$10'000$
$3-\frac{1}{1'000}$	$1'000'000$

Idea:

$$x \longrightarrow 3 \implies f(x) \longrightarrow +\infty$$

$$f(x) = \frac{1}{x^2 - 6x + 9}$$



DEF.: $f: A \longrightarrow \mathbb{R}$, $x_0 \in D(A)$

Si dice che $\lim_{x \to x_0} f(x) = +\infty \; (-\infty)$

se:

$\forall M \in \mathbb{R}, \; \exists \delta = \delta(x_0, M) > 0$:

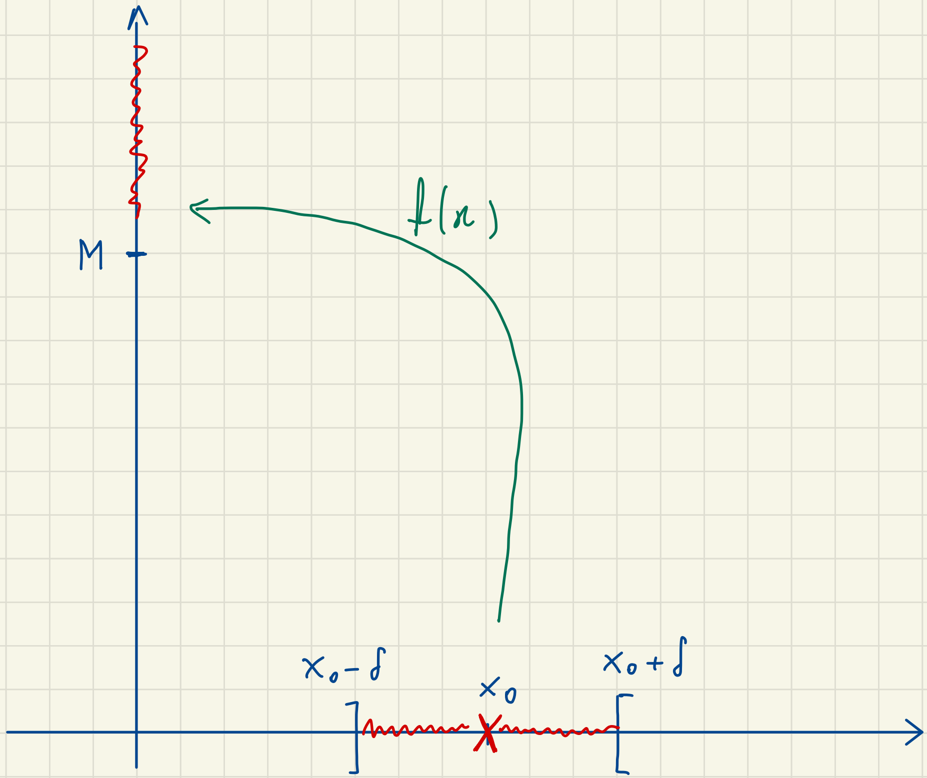
$\forall x \in A : \; 0 < |x - x_0| < \delta$

$\Longrightarrow \; f(x) > M \quad (f(x) < M)$

In tal caso la retta $x = x_0$ si dice ASINTOTO VERTICALE.

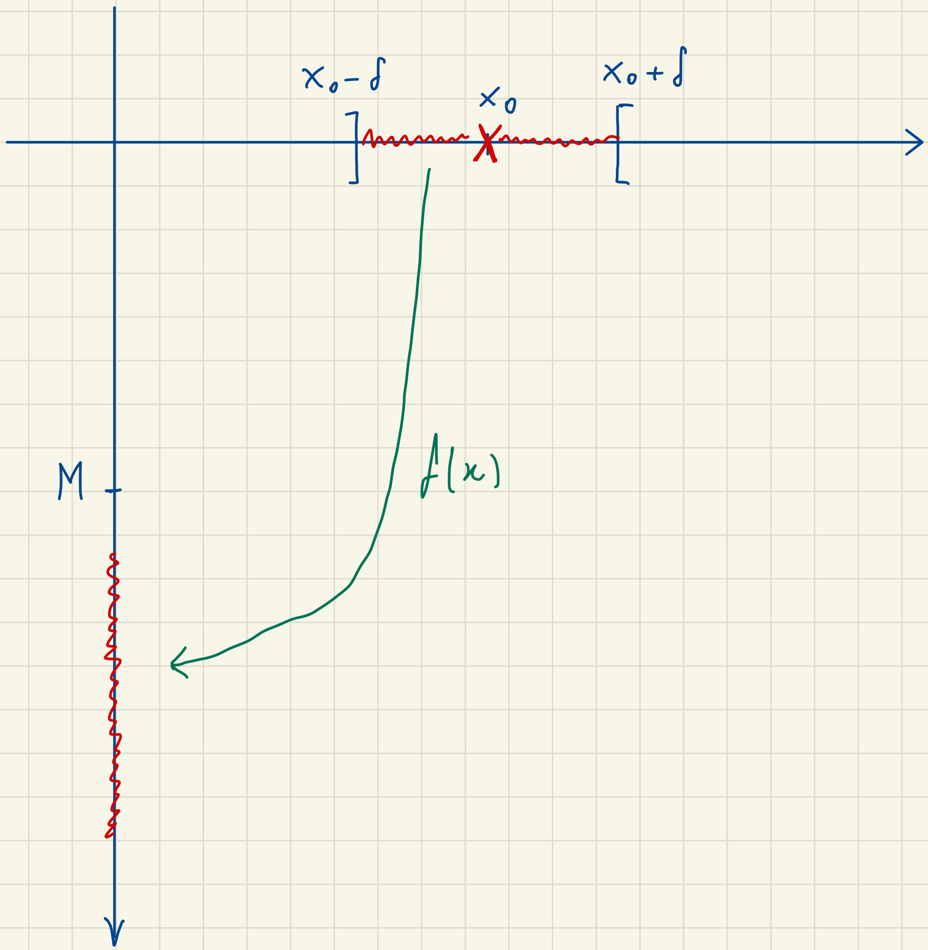
$$\lim_{x \to x_0} f(x) = +\infty$$

$\forall M \in \mathbb{R} : \exists \delta > 0 :$

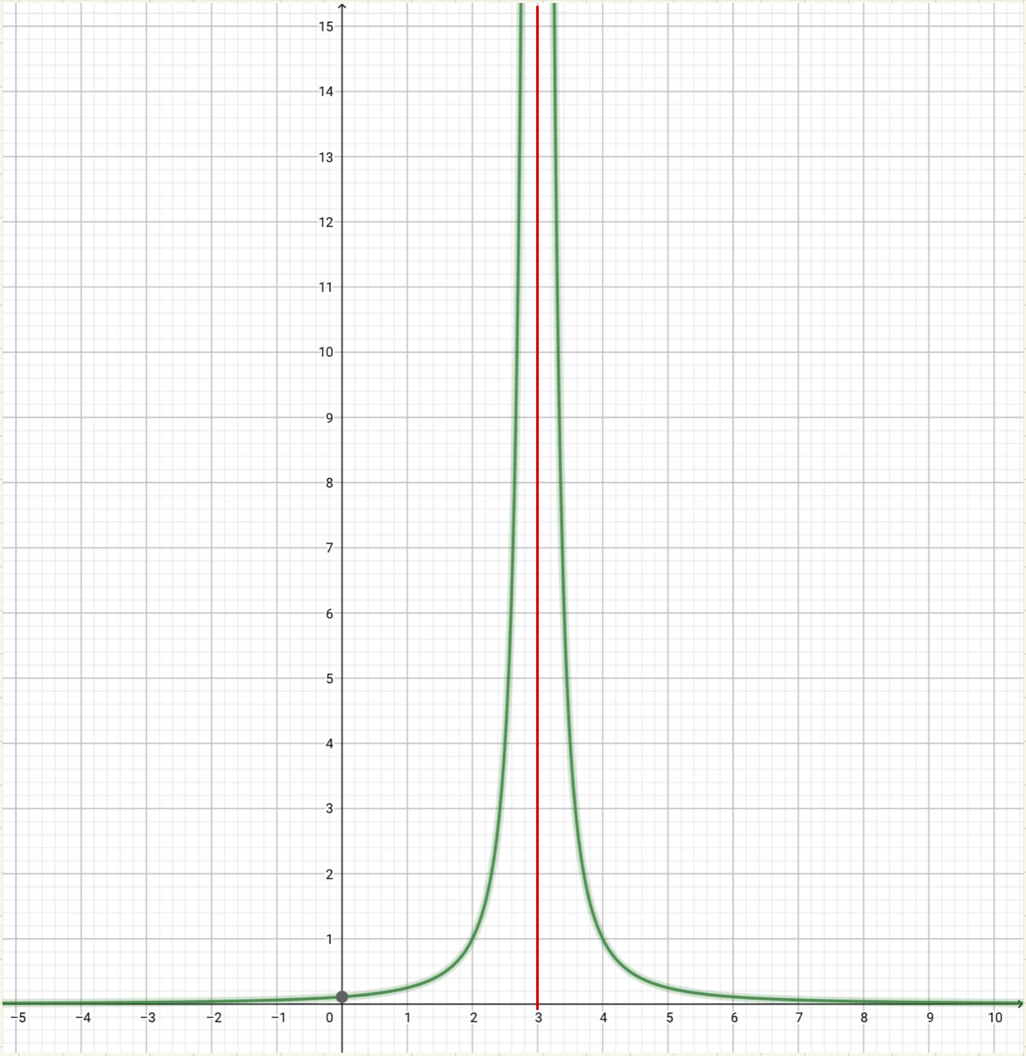


$$\lim_{x \to x_0} f(x) = -\infty$$

$\forall\, M \in \mathbb{R} : \exists\, \delta > 0 :$



$$\lim_{x \to 3} \frac{1}{x^2 - 6x + 9} = +\infty$$



$x = 3$ asintoto verticale

Dimostriamo che

$$\lim_{x \to 3} \frac{1}{x^2 - 6x + 9} = +\infty$$

$$\frac{1}{x^2 - 6x + 9} > M \qquad x \neq 3$$

$$\frac{1}{x^2-6x+9} = \frac{1}{(x-3)^2}$$

$$\iff (x-3)^2 < \frac{1}{M}$$

$$\iff -\frac{1}{\sqrt{M}} < x - 3 < \frac{1}{\sqrt{M}} \qquad (x \neq 3)$$

$$\iff 3 - \frac{1}{\sqrt{M}} < x < 3 + \frac{1}{\sqrt{M}} \qquad (x \neq 3)$$

$$\delta = \frac{1}{\sqrt{M}}$$

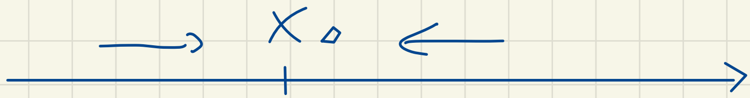
Esercizio:

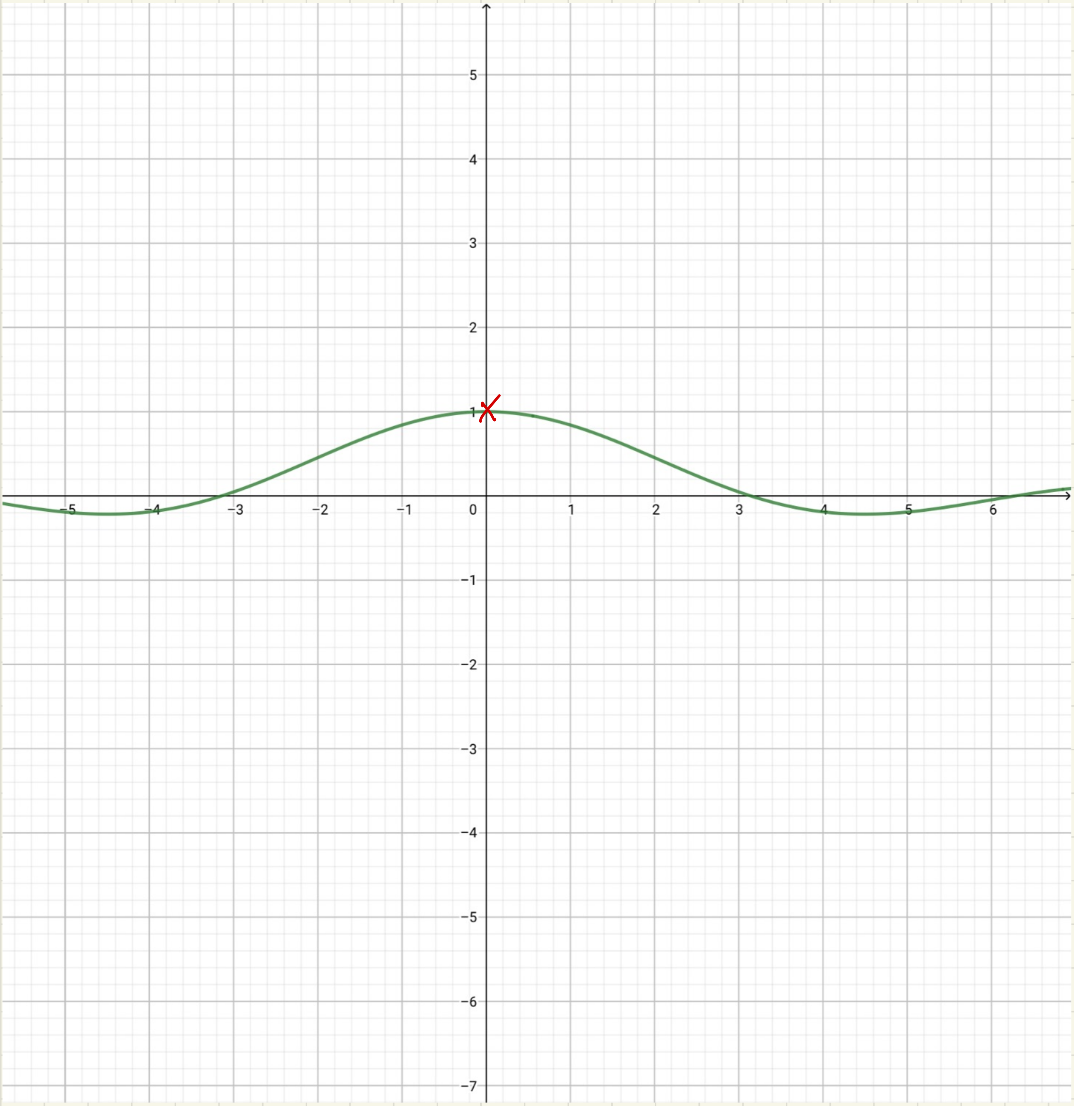
Provare che:

$$\lim_{x \to 1} \frac{1}{x^2 - 2x + 1} = +\infty$$

$$\lim_{x \to 2} \frac{1}{4x - x^2 - 4} = -\infty$$

$$\lim_{x \to x_0} f(x) = \begin{cases} \ell \in \mathbb{R} \\ +\infty \\ -\infty \end{cases}$$





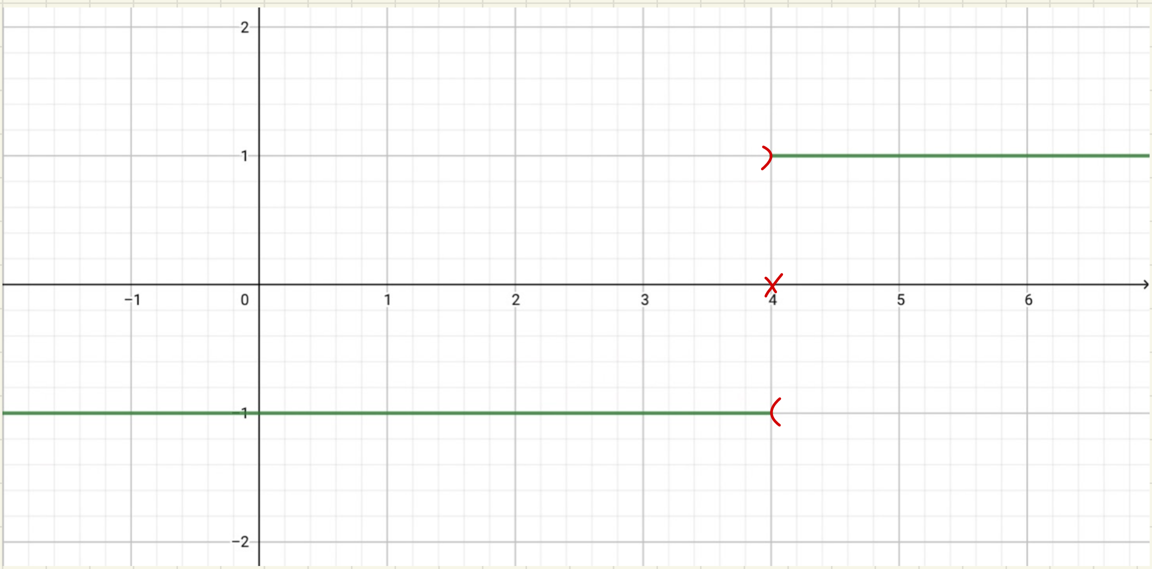
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

LIMITE DESTRO E SINISTRO:

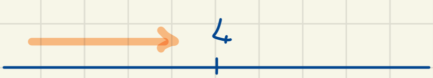
Esempio:

$$f(x) = \frac{x-4}{|x-4|} = \begin{cases} 1 & \text{se } x > 4 \\ -1 & \text{se } x < 4 \end{cases}$$

$$D(f) = \mathbb{R} \setminus \{4\}$$



$f(x) \longrightarrow 1$



$f(x) \longrightarrow -1$

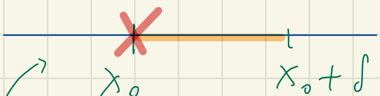
Talvolta è necessario distinguere come ci si avvicina a 4 –

DEF. (limite destro, sinistro; caso finito)

$f : A \longrightarrow \mathbb{R}$,

x_0 punto di accumulazione di A

$\ell \in \mathbb{R}$



$$\lim_{x \to x_0^+} f(x) = \ell \iff \forall \varepsilon > 0, \ \exists \, \delta = \delta(x_0, \varepsilon) > 0 : \forall x \in A : \ x_0 < x < x_0 + \delta \implies |f(x) - \ell| < \varepsilon$$

(limite destro)

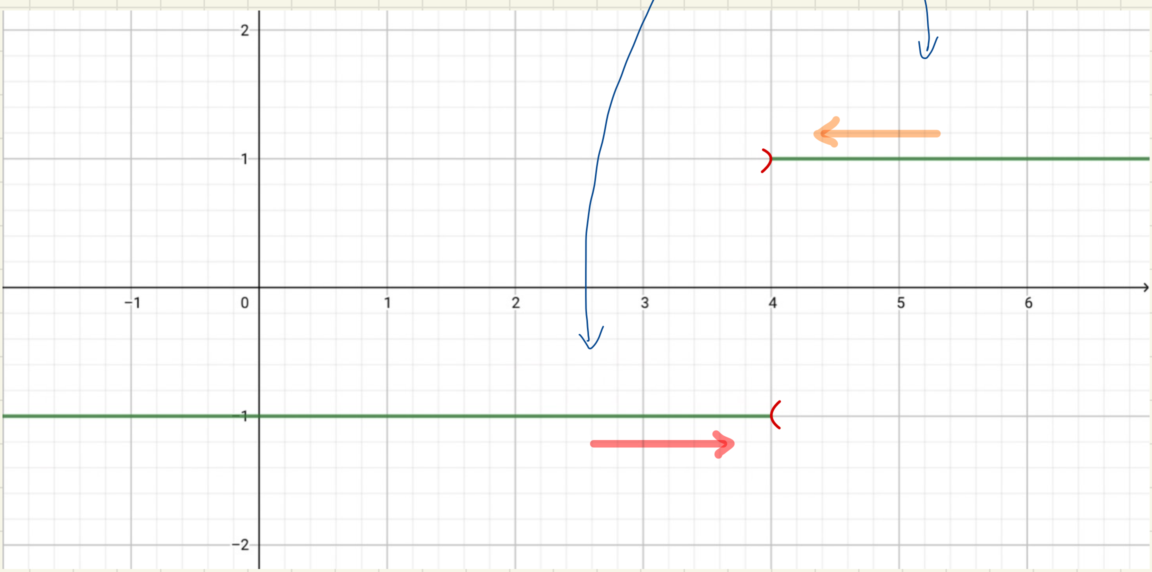
$$\lim_{x \to x_0^-} f(x) = \ell \iff \forall \varepsilon > 0, \ \exists \, \delta = \delta(x_0, \varepsilon) > 0 : \forall x \in A : \ x_0 - \delta < x < x_0 \implies |f(x) - \ell| < \varepsilon$$

(limite sinistro)

Nell' esempio precedente:

$$\lim_{x \to 4^-} \frac{x-4}{|x-4|} = -1$$

$$\lim_{x \to 4^+} \frac{x-4}{|x-4|} = 1$$



DEF. (limite destro, sinistro; caso infinito)

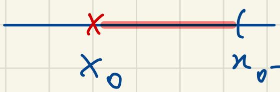
$f : A \longrightarrow \mathbb{R}$,

x_0 punto di accumulazione di A

$$\lim_{x \to x_0^+} f(x) = +\infty \;\; (-\infty) \iff \forall M \in \mathbb{R},\ \exists\, \delta = \delta(x_0, M) > 0 :$$

$$\forall x \in A : \quad x_0 < x < x_0 + \delta$$

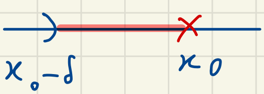
$$\implies \quad f(x) > M \quad (f(x) < M)$$



$$\lim_{x \to x_0^-} f(x) = +\infty \;\; (-\infty) \iff \forall M \in \mathbb{R},\ \exists\, \delta = \delta(x_0, M) > 0 :$$

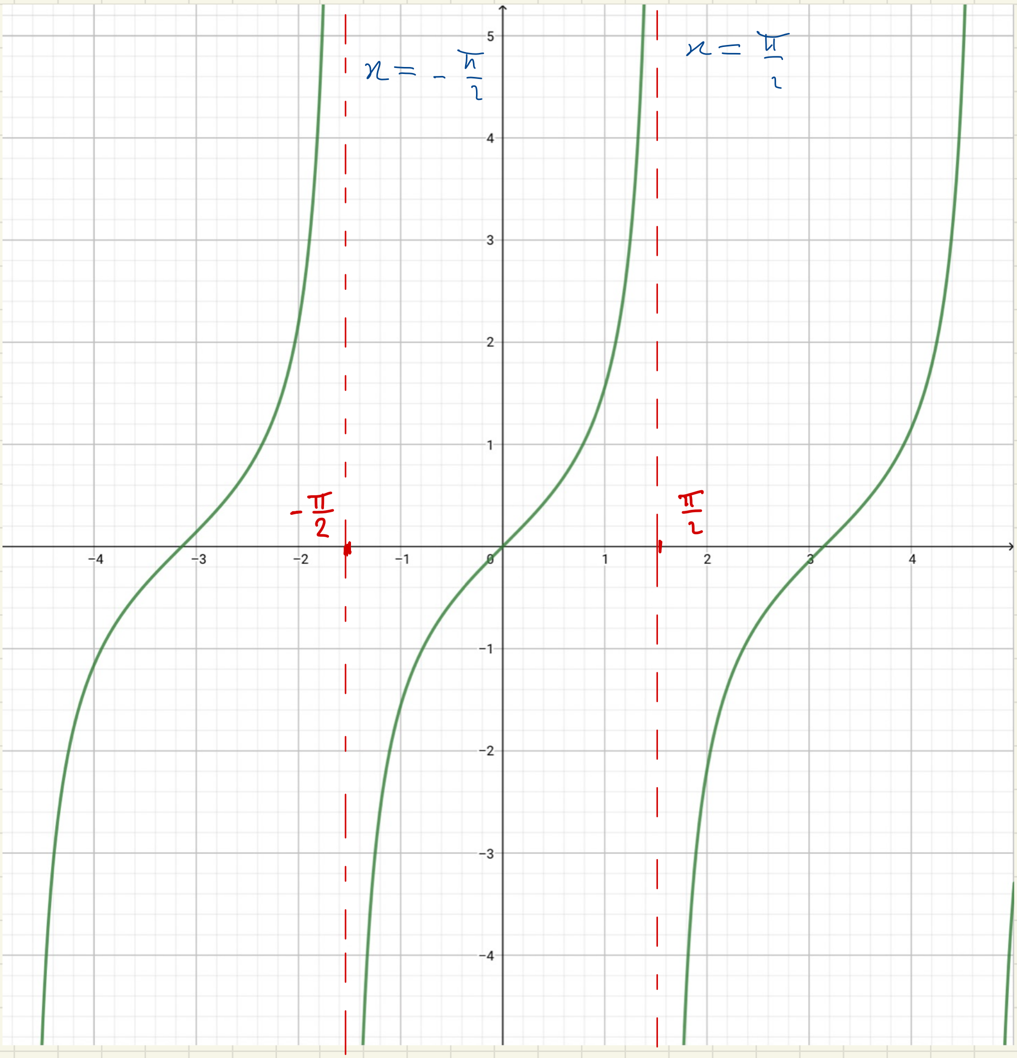
$$\forall x \in A : \quad x_0 - \delta < x < x_0$$

$$\implies \quad f(x) > M \quad (f(x) < M)$$



$x = x_0$ si dice ASINTOTO VERTICALE

Exemplo: $y = \operatorname{tg} x$



$$\lim_{x \to \frac{\pi}{2}^-} \operatorname{tg} x = +\infty \qquad \lim_{x \to \frac{\pi}{2}^+} \operatorname{tg} x = -\infty$$

$$\lim_{x \to -\frac{\pi}{2}^-} \operatorname{tg} x = +\infty \qquad \lim_{x \to -\frac{\pi}{2}^+} \operatorname{tg} x = -\infty$$

OSS. :

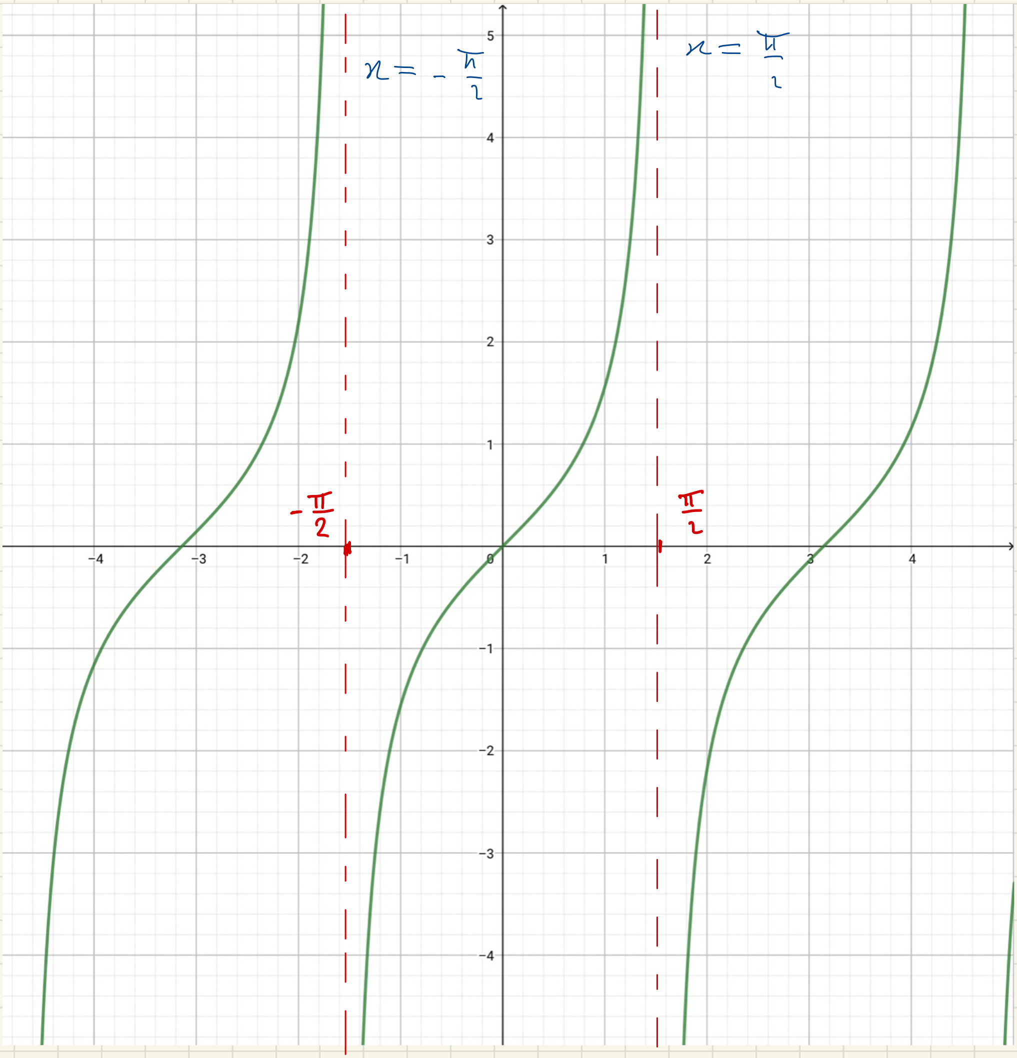
$$f : A \longrightarrow \mathbb{R}, \qquad x_0 \in D(A)$$

$$L \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$$

Allora:

$$\lim_{x \to x_0} f(x) = L \iff \begin{cases} 1) \ \exists \lim_{x \to x_0^-} f(x), \ \lim_{x \to x_0^+} f(x) \\ 2) \ \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = L \end{cases}$$

Ejemplo: $y = \operatorname{tg} x$



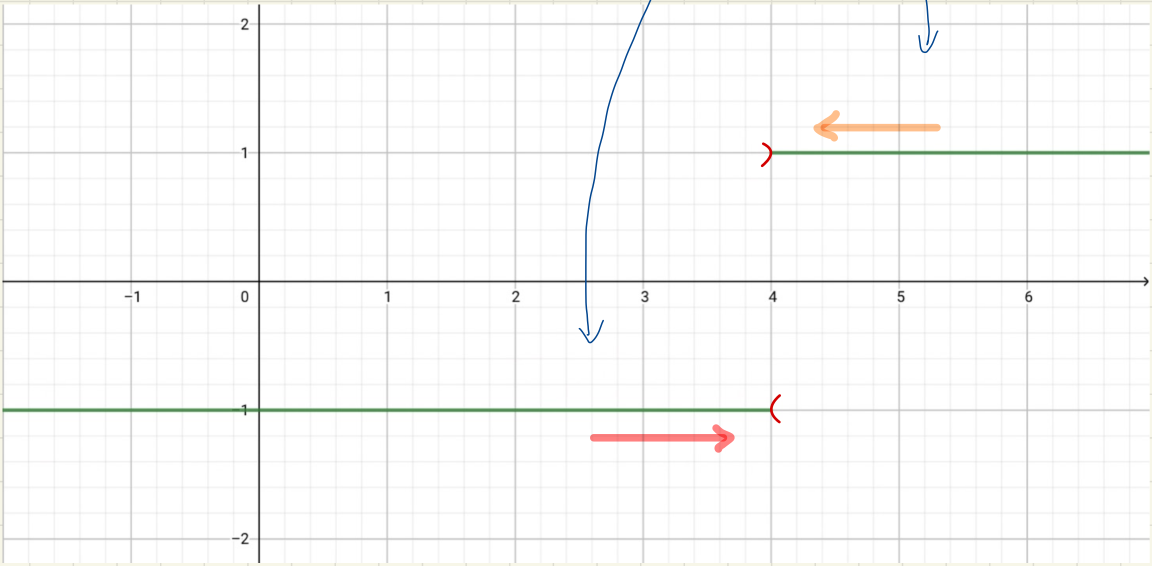
$$\nexists \lim_{x \to \frac{\pi}{2}} \operatorname{tg} x$$

$$\nexists \lim_{x \to -\frac{\pi}{2}} \operatorname{tg} x$$

Nell' esempio precedente:

$$\lim_{x \to 4^-} \frac{x-4}{|x-4|} = -1$$

$$\lim_{x \to 4^+} \frac{x-4}{|x-4|} = 1$$



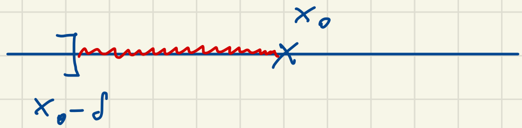
$$\nexists \lim_{x \to 4} \frac{x-4}{|x-4|}$$

PROP. (Forma $\frac{1}{0}$) : $\lim \frac{1}{f(x)}$

$f : A \longrightarrow \mathbb{R}$, $x_0 \in D(A)$

$\exists\, \delta > 0$:

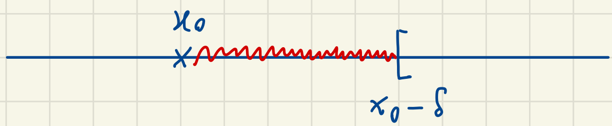
I $f(x) > 0$ (< 0) $\quad \forall x \in A : \quad x_0 - \delta < x < x_0$



$\lim\limits_{x \to x_0^-} f(x) = 0$

Allora $\lim\limits_{x \to x_0^-} \frac{1}{f(x)} = +\infty \quad (-\infty)$

II $f(x) > 0$ (< 0) $\quad \forall x \in A : \quad x_0 < x < x_0 + \delta$

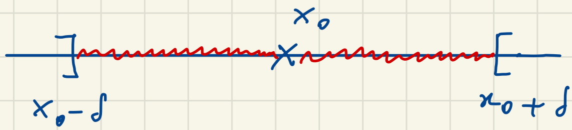


$\lim\limits_{x \to x_0^+} f(x) = 0$

Allora $\lim\limits_{x \to x_0^+} \frac{1}{f(x)} = +\infty \quad (-\infty)$

$x \neq x_0$

III $f(x) > 0$ (< 0) $\quad \forall x \in A: \; x_0 - \delta < x < x_0 + \delta$



$$\lim_{x \to x_0} f(x) = 0$$

Allora $\displaystyle \lim_{x \to x_0} \frac{1}{f(x)} = +\infty \quad (-\infty)$

Esempi:

$x - 2 \qquad - \quad - \quad - \quad \overset{2}{|} \quad + \quad + \quad +$

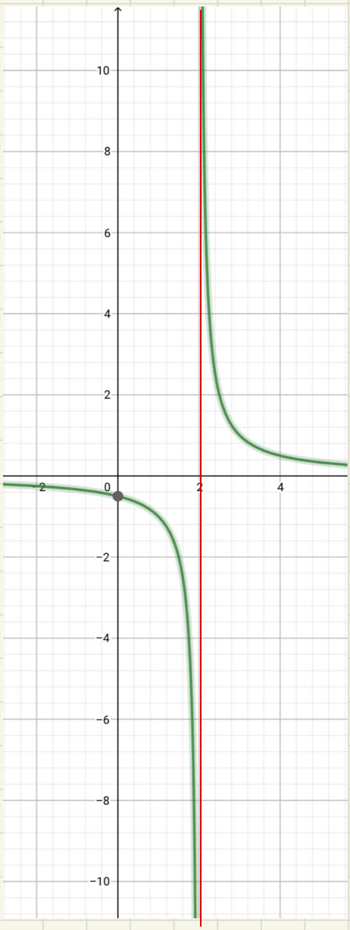
1.

$$\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$$

$x - 2 > 0$

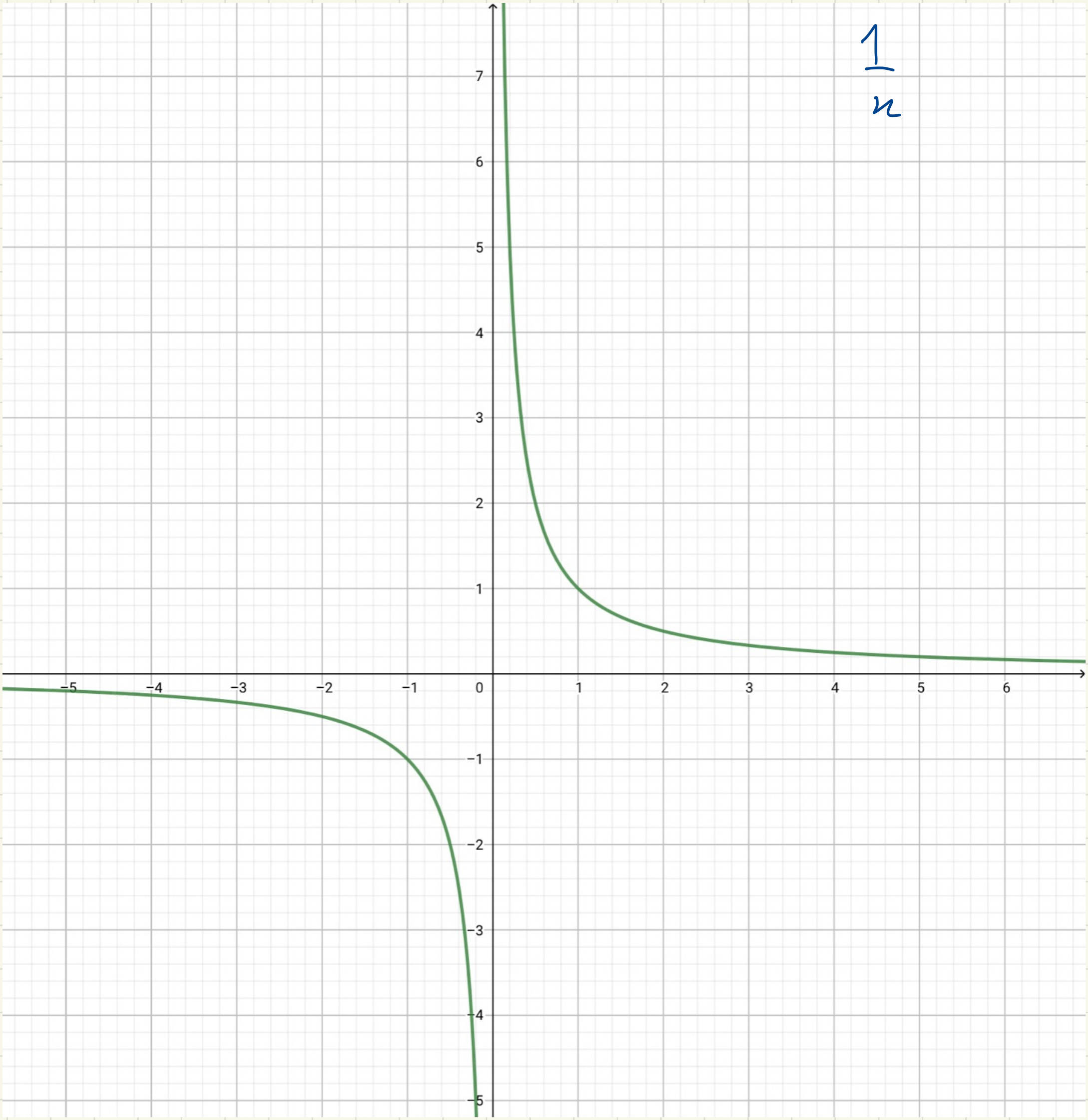
$x > 2$

$$\lim_{x \to 2^+} \frac{1}{x-2} = +\infty$$



$x = 2$ ASINTOTO VERTICALE

$$\nexists \lim_{x \to 2} \frac{1}{x-2}$$



Esiste il limite: $\lim_{x \to 0} \frac{1}{x}$?

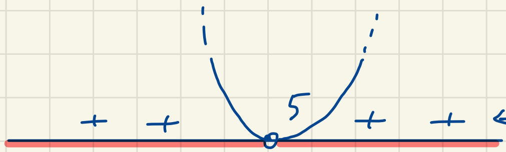
$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

NO

2)

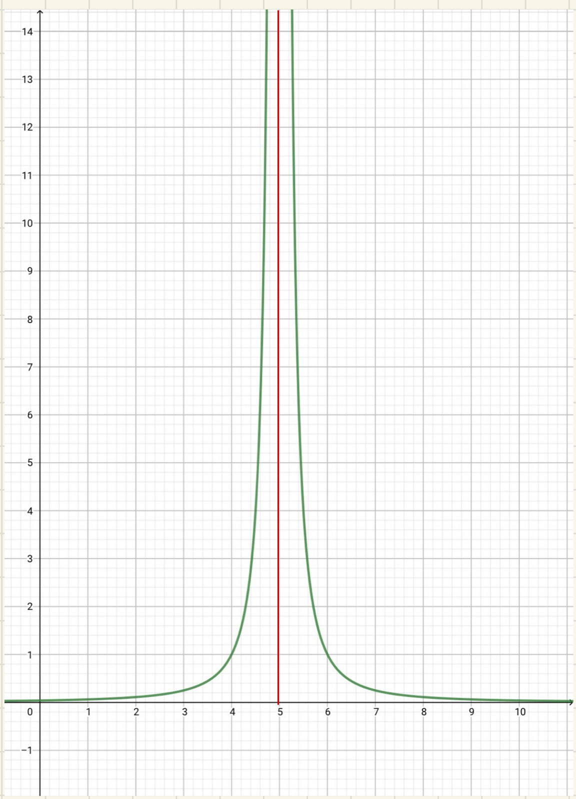
$$\lim_{x \to 5} \frac{1}{x^2 - 10x + 25}$$



$$x^2 - 10x + 25 = 0$$
$$\downarrow$$
$$x = 5$$

$$\Rightarrow \lim_{x \to 5^-} \frac{1}{x^2 - 10x + 25} = +\infty$$

$$\lim_{x \to 5^+} \frac{1}{x^2 - 10x + 25} = +\infty$$



$$\lim_{x \to 5} \frac{1}{x^2 - 10x + 25} = +\infty$$

$x = 5$

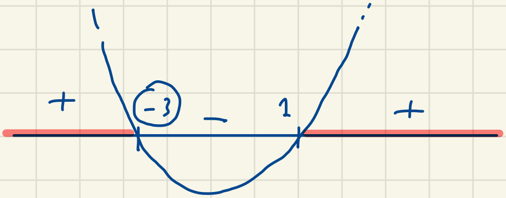
Asintoto

verticale

3

$$\lim_{x \to -3^{\pm}} \frac{1}{x^2+2x-3} = \,!$$

$$x^2+2x-3=0 \longrightarrow x_{1,2} = \frac{-2 \pm 4}{2} = \begin{cases} -3 \\ 1 \end{cases}$$



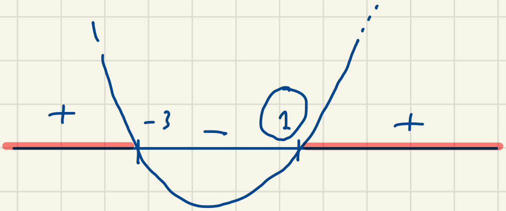
$\leftarrow x^2+2x-3$

$$\lim_{x \to -3^-} \frac{1}{x^2+2x-3} = +\infty$$

$$\lim_{x \to -3^+} \frac{1}{x^2+2x-3} = -\infty$$

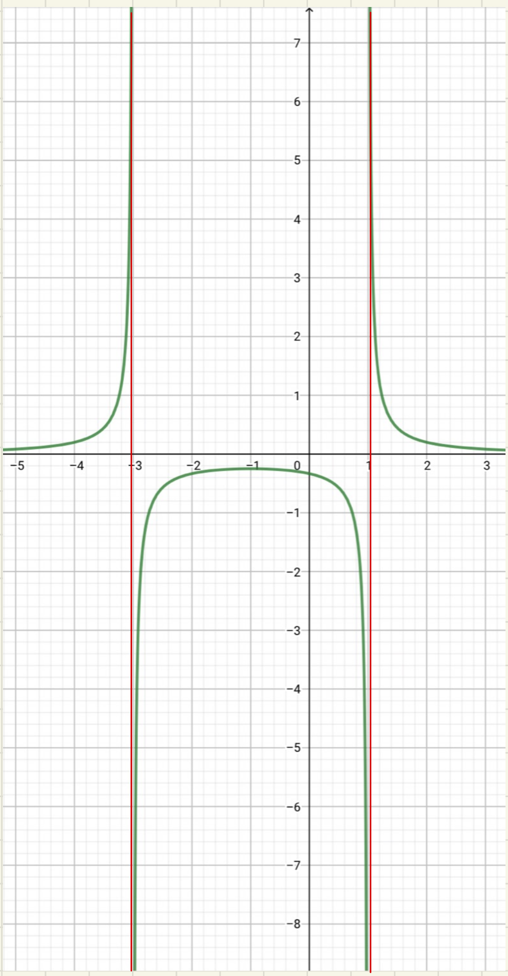
$$\Longrightarrow \nexists \lim_{x \to -3} \frac{1}{x^2+2x-3}$$

$$\lim_{x \to 1^{\pm}} \frac{1}{x^2+2x-3}$$



$$\lim_{x \to 1^-} \frac{1}{x^2+2x-3} = -\infty \qquad \Rightarrow \quad \nexists \lim_{x \to 1}$$

$$\lim_{x \to 1^+} \frac{1}{x^2+2x-3} = +\infty$$



$x = -3$

$x = 1$

asintoti verticali

$$\lim_{x \to 0} \frac{1}{x^2+2x-3} =$$

$$= \frac{1}{\lim\limits_{x \to 0} \left(x^2+2x-3\right)} = \frac{1}{-3} = -\frac{1}{3}$$

LIMITE: caso all' infinito

DEF. $(x \to +\infty)$:

$$f : A \longrightarrow \mathbb{R}$$

$$\sup A = +\infty$$

Si dice che:

$y = \ell$
ASINTOTO ORIZZONTALE

$$\lim_{x \to +\infty} f(x) = \begin{cases} \ell \in \mathbb{R} & (1) \\ +\infty & (2) \\ -\infty & (3) \end{cases} \quad \text{se:}$$

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(\varepsilon) > 0 : \forall x \in A : x > \delta$$

$$\Longrightarrow \begin{cases} (1) & |f(x) - \ell| < \varepsilon \\ (2) & f(x) > \varepsilon \\ (3) & f(x) < -\varepsilon \end{cases}$$

DEF. $(x \to -\infty)$:

$$f : A \longrightarrow \mathbb{R}$$

$$\inf A = -\infty$$

Si dice che:

$$\lim_{x \to -\infty} f(x) = \begin{cases} \ell \in \mathbb{R} & (1) \\ +\infty & (2) \\ -\infty & (3) \end{cases} \quad \text{se :}$$

$y = \ell$ ASINTOTO ORIZZONTALE

$$\forall \varepsilon > 0, \ \exists \delta = \delta(\varepsilon) > 0 : \forall x \in A : x < -\delta :$$

$$\Longrightarrow \begin{cases} (1) & |f(x) - \ell| < \varepsilon \\ (2) & f(x) > \varepsilon \\ (3) & f(x) < -\varepsilon \end{cases}$$

Esempio:

$$\lim_{x \to +\infty} x^2 = +\infty$$

$$\forall \varepsilon > 0, \ \exists \delta > 0 : \forall x \in \mathbb{R} : \quad x > \delta$$

$$\Longrightarrow \quad x^2 > \varepsilon$$

$$x^2 > \varepsilon \iff x < -\sqrt{\varepsilon} \ \vee \ x > \sqrt{\varepsilon}$$

Scegliamo $\delta = \sqrt{\varepsilon}$:

$$x > \delta = \sqrt{\varepsilon} \Longrightarrow x^2 > \varepsilon$$

Fine

Esercizio:

Provare che $\displaystyle\lim_{x \to -\infty} x^2 = +\infty$.

$$\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$$

$$\downarrow$$

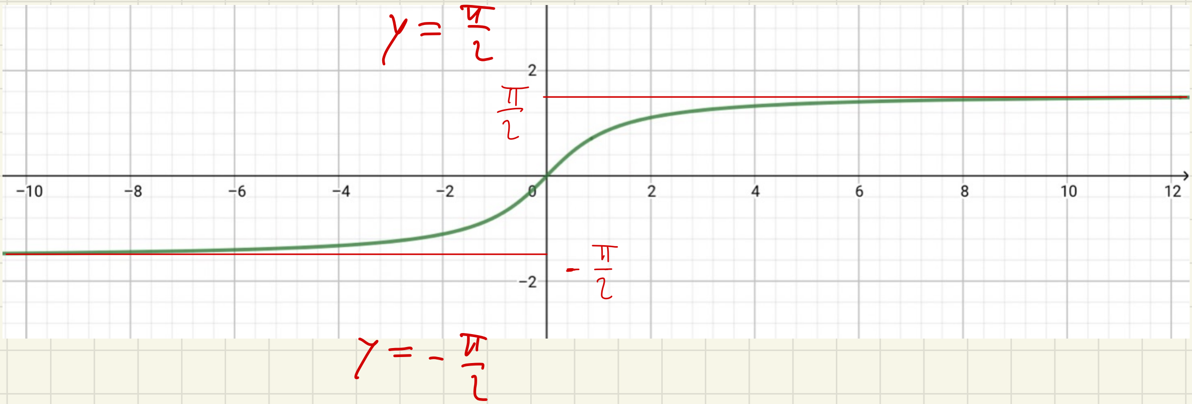
$$\lim_{x \to \pm\infty} \frac{c}{x^n} = \underbrace{\lim_{x \to \pm\infty} c}_{= \, c} \cdot \underbrace{\lim_{x \to \pm\infty} \frac{1}{x^n}}_{\to \, 0}$$

$$= 0$$

Esempi:

$$\lim_{x \to -\infty} \arctg x = -\frac{\pi}{2}$$

$$\lim_{x \to +\infty} \arctg x = \frac{\pi}{2}$$



$y = \frac{\pi}{2}$

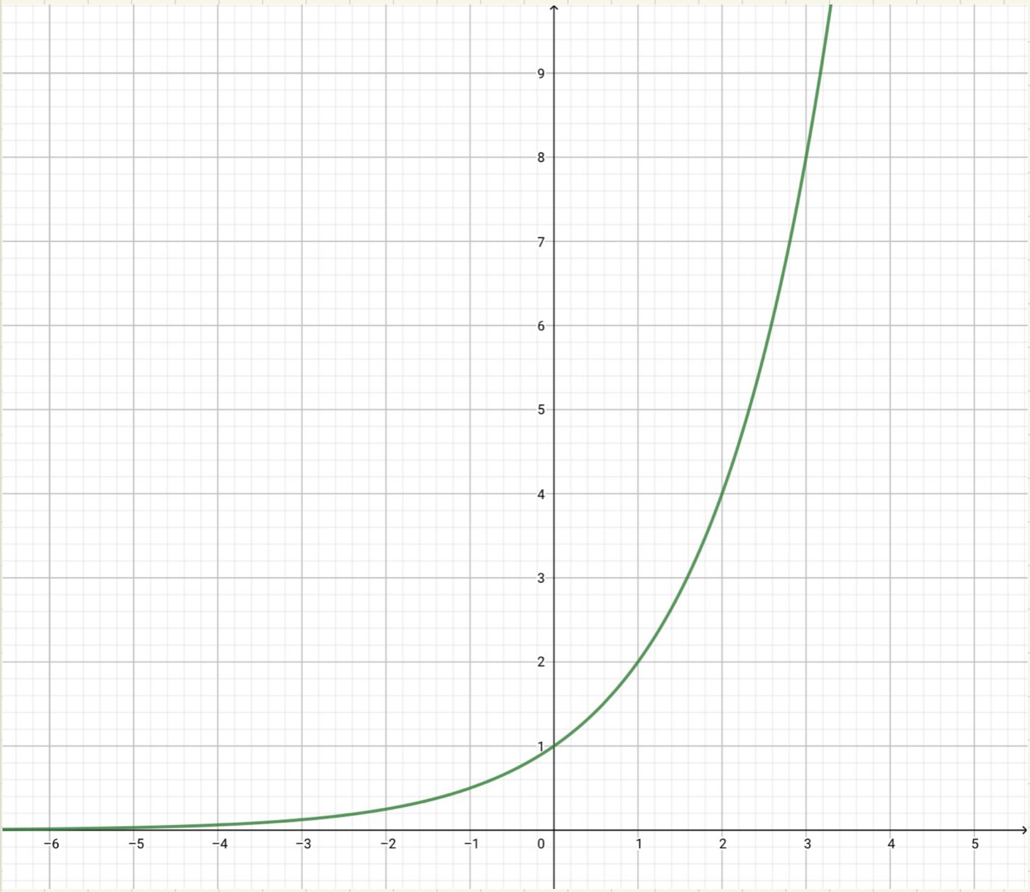
$y = -\frac{\pi}{2}$

ASINTOTI ORIZZONTALI

$a \in \mathbb{R} : \quad a > 1$

$$\lim_{x \to -\infty} a^x = 0^+$$

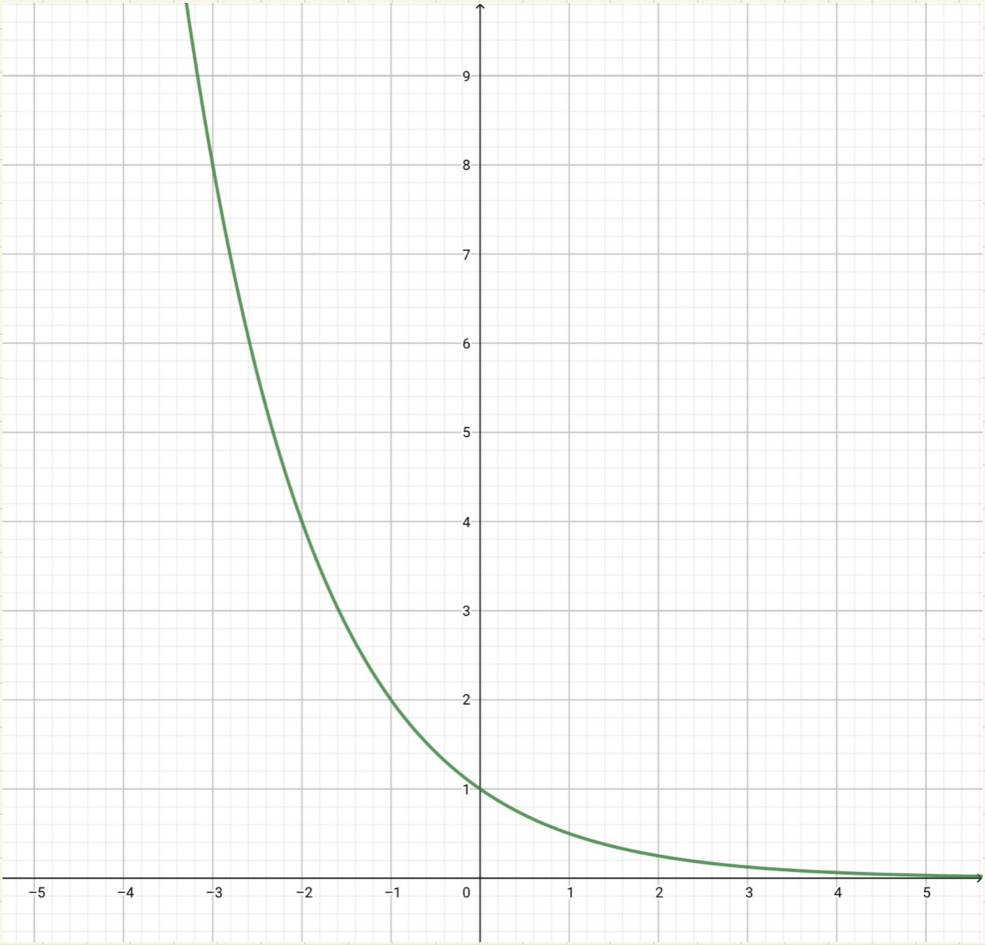
$$\lim_{x \to +\infty} a^x = +\infty$$



$a \in \mathbb{R}: \quad 0 < a < 1$

$$\lim_{x \to -\infty} a^x = +\infty$$

$$\lim_{x \to +\infty} a^x = 0^+$$

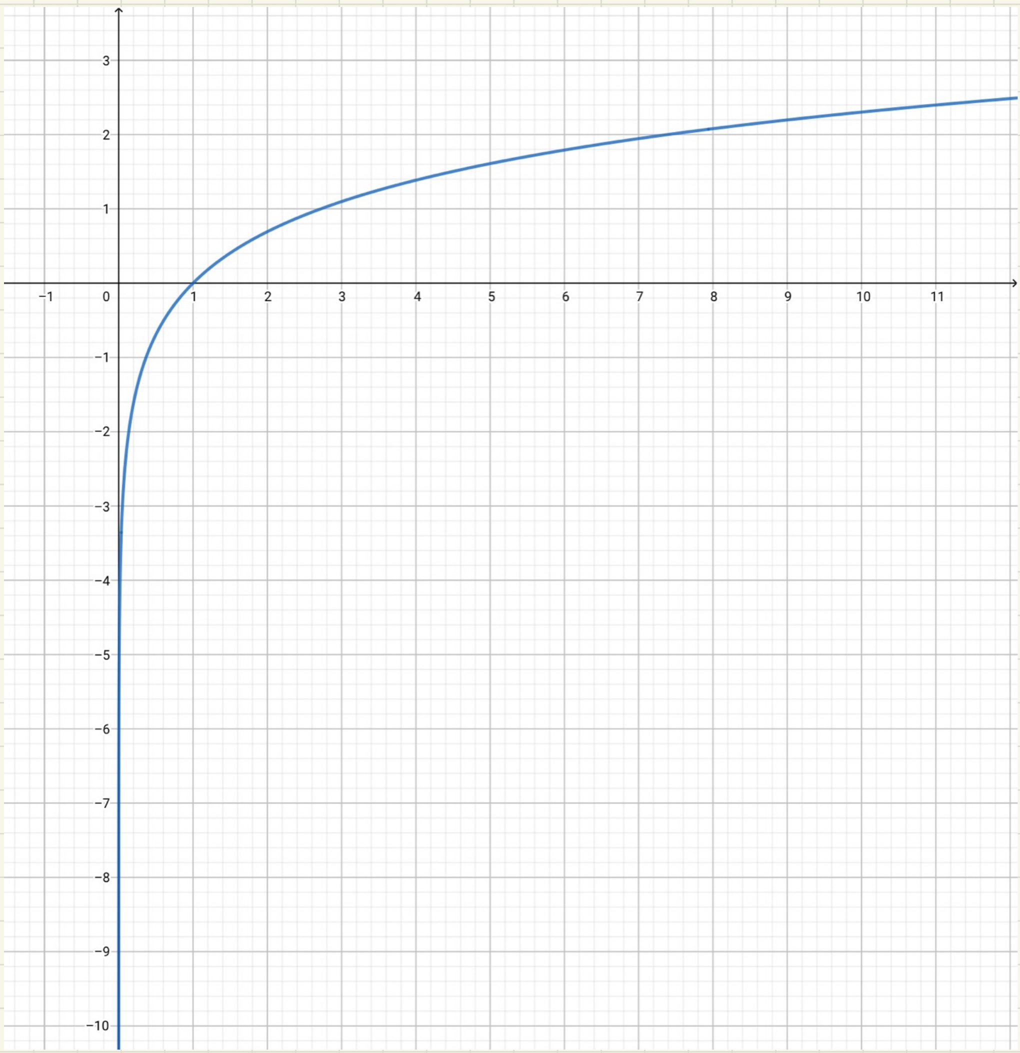


$a \in \mathbb{R}: \quad a > 1$

$$\lim_{x \to +\infty} \log_a x = +\infty$$

$$\lim_{x \to 0^+} \log_a x = -\infty$$

$x = 0$ ASINTOTO VERTICALE

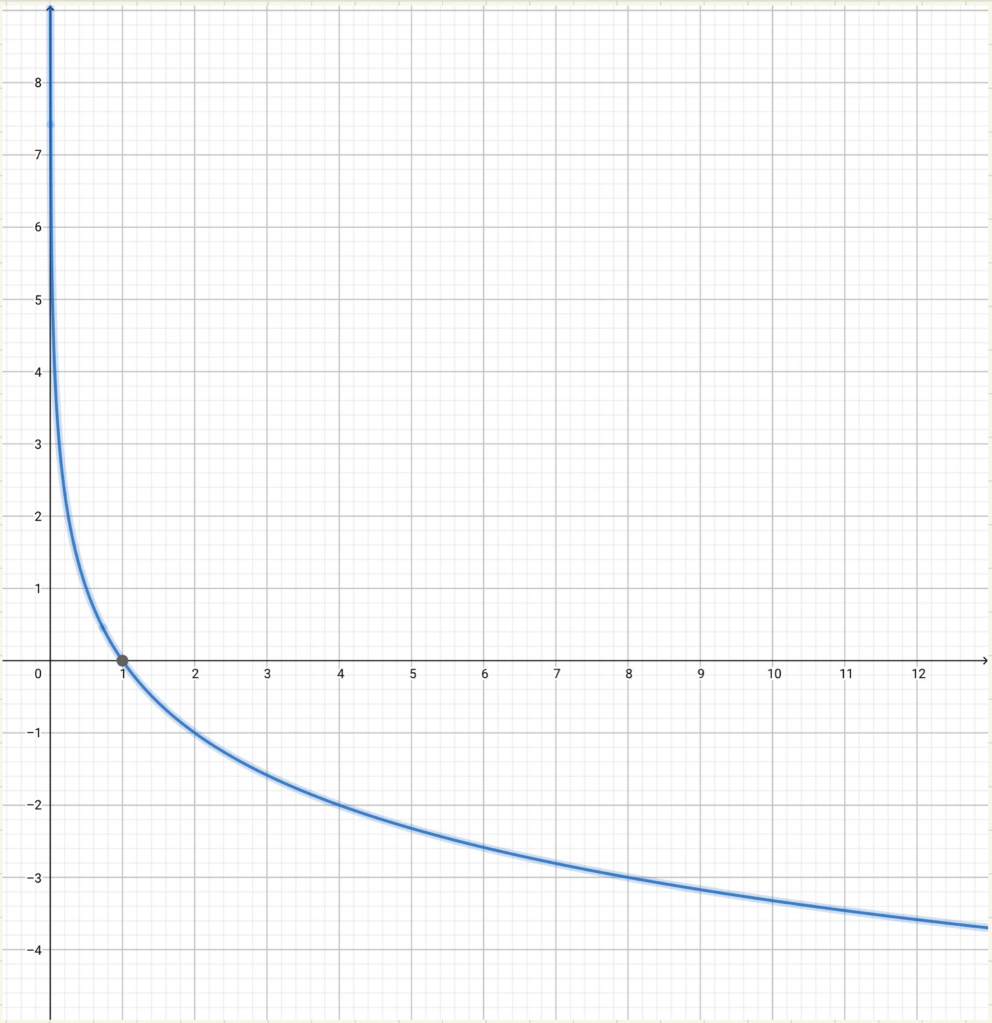


$a \in \mathbb{R}: \quad 0 < a < 1$

$$\lim_{x \to +\infty} \log_a x = -\infty$$

$$\lim_{x \to 0^+} \log_a x = +\infty$$

$x = 0$
ASINTOTO
VERTICALE



SCHEMA RIASSUNTIVO:

$$\lim_{\substack{x \to x_0 \ (I) \\ x \to x_0^+ \ (II) \\ x \to x_0^- \ (III) \\ x \to +\infty \ (IV) \\ x \to -\infty \ (V)}} f(x) = \begin{cases} \ell \in \mathbb{R} & (1) \\ +\infty & (2) \\ -\infty & (3) \end{cases}$$

$$\forall \varepsilon > 0, \ \exists \delta > 0 : \forall x \in D(f) : \begin{cases} (I) & 0 < |x - x_0| < \delta \\ (II) & x_0 < x < x_0 + \delta \\ (III) & x_0 - \delta < x < x_0 \\ (IV) & x > \delta \\ (V) & x < -\delta \end{cases}$$

$$\Longrightarrow \begin{cases} (1) & |f(x) - \ell| < \varepsilon \\ (2) & f(x) > \varepsilon \\ (3) & f(x) < -\varepsilon \end{cases}$$

EJEMPLO:

$$\lim_{x \to 5^-} f(x) = -4$$

$$\forall \varepsilon > 0, \ \exists \delta > 0 : \ \forall x \in D(f): \ 5 - \delta < x < 5$$

$$\Longrightarrow \quad |f(x) - (-4)| < \varepsilon$$

$$|f(x) + 4| < \varepsilon$$

$$\lim_{x \to +\infty} f(x) = -\infty$$

$(\forall \varepsilon \in \mathbb{R})$

$$\cancel{\forall \varepsilon > 0}, \ \exists \delta > 0 : \ \forall x \in D(f): \ x > \delta$$

$$\Longrightarrow \quad f(x) < -\varepsilon$$

$$(f(x) < \varepsilon)$$

Vale l'algebra dei limiti vista per le successioni numeriche.

$$f, g : A \longrightarrow \mathbb{R}$$

allora:

$$\cancel{\begin{pmatrix} +\infty - \infty \\ -\infty + \infty \end{pmatrix}}$$

$$\lim \left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x)$$

$$\lim \left(f(x) \cdot g(x)\right) = \lim f(x) \cdot \lim g(x) \qquad \cancel{\boxed{0 \cdot (\pm\infty)}} \quad \cancel{\boxed{\pm\infty \cdot 0}}$$

$$\lim \frac{1}{g(x)} = \frac{1}{\lim g(x)} \qquad \cancel{\frac{1}{0}}$$

$$\lim \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \qquad \cancel{\frac{0}{0}}, \ \cancel{\frac{\pm\infty}{\pm\infty}}$$

(con la convenzione che il secondo membro deve avere significato)

Forme indeterminate:

$+\infty - \infty$, $-\infty + \infty$, $0 \cdot (\pm\infty)$,

$(\pm\infty) \cdot 0$, $\frac{\pm\infty}{\pm\infty}$, $\frac{0}{0}$, $\frac{\ell}{0}$,

$1^{+\infty}$, $1^{-\infty}$, 0^{0}

Conseguenze:

1. $$\lim_{x\to+\infty} x = +\infty \quad (\text{immediato})$$

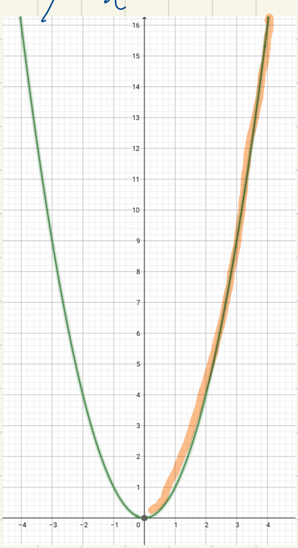
$$\lim_{x\to+\infty} x^2 = \lim_{x\to+\infty} x \cdot \lim_{x\to+\infty} x = +\infty$$

$$\lim_{x\to+\infty} x^3 = \lim_{x\to+\infty} x^2 \cdot \lim_{x\to+\infty} x = +\infty$$

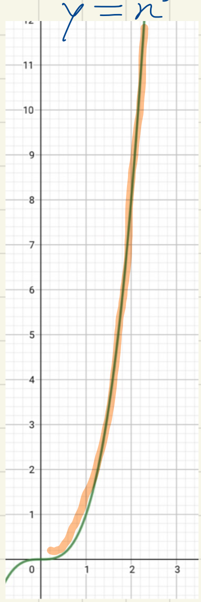
. . . .

$$\lim_{x\to+\infty} x^n = +\infty$$

$y=x^2$



$y=x^3$



$y=x^4$

$$+\infty \cdot +\infty = +\infty$$

$$\lim \underset{+\infty}{\underset{\downarrow}{f(x)}} \cdot \underset{+\infty}{\underset{\downarrow}{g(x)}} = +\infty \quad \checkmark$$

<u>NO</u>

$$\lim_{x \to +\infty} \cancel{f(x) \cdot g(x) = +\infty \cdot (+\infty) = +\infty}$$

②

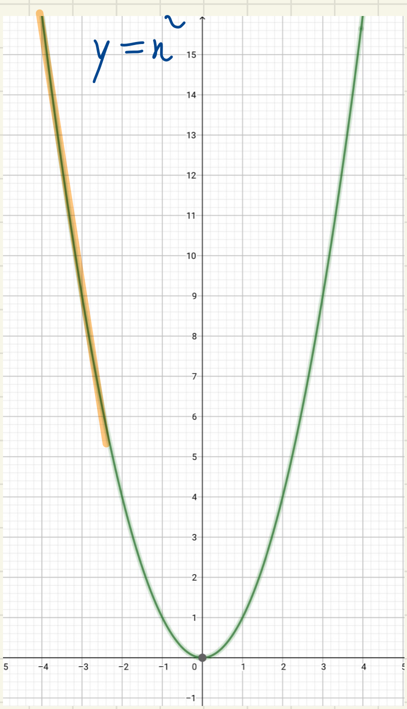
$$\lim_{x\to-\infty} x = -\infty \qquad \text{(immediato)}$$

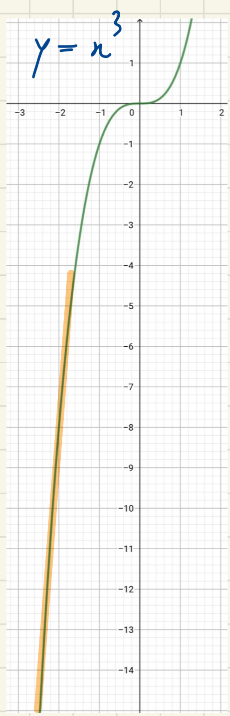
$$\lim_{x\to-\infty} x^2 = \lim_{x\to-\infty} x \cdot \lim_{x\to-\infty} x = +\infty$$

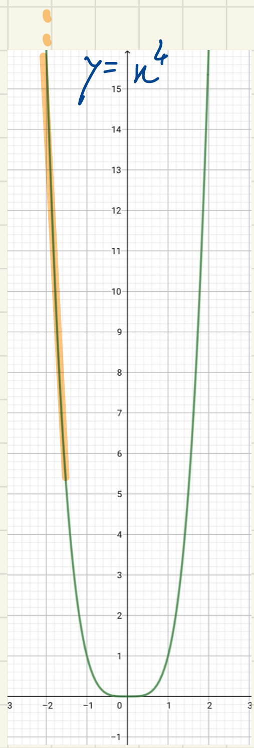
$$\lim_{x\to-\infty} x^3 = \lim_{x\to-\infty} x^2 \cdot \lim_{x\to-\infty} x = -\infty$$

. . . . . .

$$\lim_{x\to-\infty} x^n = \begin{cases} +\infty & \text{se } n \text{ è pari} \\ -\infty & \text{se } n \text{ è dispari} \end{cases}$$







(3)

$$\lim_{x \to +\infty} \underbrace{\left(4x^3 - 2x + 4\right)}_{\substack{\| \\ x^3 \cdot \boxed{\left(4 - \frac{2}{x^2} + \frac{4}{x^3}\right)}}} = +\infty$$

$$x^3 \to +\infty \qquad \left(4 - \frac{2}{x^2} + \frac{4}{x^3}\right) \to 4$$

$$\lim_{x \to -\infty} \underbrace{\left(4x^3 - 2x + 4\right)}_{\substack{\| \\ x^3 \cdot \underbrace{\left(4 - \frac{2}{x^2} + \frac{4}{x^3}\right)}}} = -\infty$$

$$x^3 \to -\infty \qquad \left(4 - \frac{2}{x^2} + \frac{4}{x^3}\right) \to 4 > 0$$

$$\left(\lim_{x \to -\infty} \frac{2}{x^2} = 0 \right)$$

$$\lim_{x \to +\infty} \underbrace{\left(4x^3 - 2x^2 + x\right)}$$

$$x\left(4x^2 - 2x + 1\right)$$

$$x \to +\infty \qquad 4x^2 \to +\infty \qquad -2x \to -\infty$$

$$\lim_{x \to -\infty} \underbrace{\left(-4x^3 - 2x + 4\right)} = +\infty$$

$$x^3 \underbrace{\left(-4 - \frac{2}{x^2} + \frac{4}{x^3}\right)}$$

$$x^3 \to -\infty \qquad \left(-4 - \frac{2}{x^2} + \frac{4}{x^3}\right) \to -4 < 0$$

4

$$p(x) = \sum_{j=0}^{n} a_j x^j \qquad (a_n > 0)$$

$$= x^n \cdot \sum_{j=0}^{n} a_j \cdot \frac{1}{x^{n-j}} =$$

$$= x^n \cdot \left(a_n + a_{n-1} \cdot \frac{1}{x} + a_{n-2} \cdot \frac{1}{x^2} + \cdots \right.$$

$$\left. \cdots + a_1 \cdot \frac{1}{x^{n-1}} + \frac{a_0}{x^n} \right)$$

n = pari

$a_n < 0$

$$\lim_{x \to +\infty} p(x) = +\infty \qquad (-\infty)$$

$$\lim_{x \to -\infty} p(x) = +\infty \qquad (-\infty)$$

n = dispari:

$a_n < 0$ $(-\infty)$ $\qquad$ $a_n < 0$ $(+\infty)$

$$\lim_{x \to +\infty} p(x) = +\infty \qquad \lim_{x \to -\infty} p(x) = -\infty$$

4' $x_0 \in \mathbb{R}$:

$$\lim_{x \to x_0} x = x_0$$ (dalla definizione di limite con $\delta = \varepsilon$)

Dall'algebra dei limiti:

$$\lim_{x \to x_0} x^2 = \underbrace{\boxed{\lim_{x \to x_0} x}}_{= x_0} \cdot \underbrace{\boxed{\lim_{x \to x_0} x}}_{= x_0} = x_0^2$$

$$\lim_{x \to x_0} x^3 = \lim_{x \to x_0} x^2 \cdot \lim_{x \to x_0} x = x_0^3$$

$$\vdots$$

$$\boxed{\lim_{x \to x_0} x^j = x_0^j}$$ $(j \in \mathbb{N})$

$\lambda \in \mathbb{R}$:

$$\lim_{x \to x_0} \lambda \cdot x^j = \lim_{x \to x_0} \lambda \cdot \lim_{x \to x_0} x^j = \lambda \cdot x_0^j$$

Possiamo così provare una importante proprietà dei polinomi:

$$p(x) = \sum_{j=0}^{n} a_j \cdot x^j$$

polinomio di grado n

Vale:

$$\lim_{x \to x_0} p(x) = p(x_0)$$

Infatti:

$$\lim_{x \to x_0} \sum_{j=0}^{n} a_j \cdot x^j =$$

$$= \sum_{j=0}^{n} \lim_{x \to x_0} \left(a_j \cdot x^j \right) =$$

$$= \sum_{j=0}^{n} a_j \cdot x_0^j = p(x_0)$$

(5)

$$\lim_{x\to+\infty} \frac{-2x^6 - x^5 + x^3 - 4}{4x^6 - 3x^2 - 5} =$$

$$= \lim_{x\to+\infty} \underbrace{\frac{x^6}{x^6}}_{\substack{\parallel \\ 1}} \cdot \boxed{\frac{-2 - \frac{1}{x} + \frac{1}{x^3} - \frac{4}{x^6}}{4 - \frac{3}{x^4} - \frac{5}{x^6}}} = -\frac{1}{2}$$

$$\downarrow$$

$$\frac{-2}{4} = -\frac{1}{2}$$

(6)

$$\lim_{x\to-\infty} \frac{x^5 + x^7 - x^2 + 1}{x^3 - 4x^4 + x^2 - 4} = +\infty$$

$$= \lim_{x\to-\infty} \underbrace{\frac{x^7}{x^4}}_{\substack{\parallel \\ x^3 \\ \downarrow \\ -\infty}} \cdot \boxed{\frac{\frac{1}{x^2} + 1 - \frac{1}{x^5} + \frac{1}{x^7}}{\frac{1}{x} - 4 + \frac{1}{x^2} - \frac{4}{x^4}}}$$

$$\downarrow$$

$$\frac{1}{-4} = -\frac{1}{4}$$

6'

$$\lim_{x \to +\infty} \frac{4x^6 - x^4}{7x^3 + 10x^9} =$$

$$= \lim_{x \to +\infty} \frac{x^6}{x^9} \cdot \boxed{\frac{4 - \frac{1}{x^2}}{\frac{7}{x^6} + 10}} = 0$$

$$\frac{x^6}{x^9} = \frac{1}{x^3} \to 0$$

$$\boxed{\frac{4 - \frac{1}{x^2}}{\frac{7}{x^6} + 10}} \to \frac{4}{10} = \frac{2}{5}$$