21. Ottobre. 2021

I PARZIALE:

22 Dicembre

ore 9:30/10:00 (inizio)

FUNZIONI GONIOMETRICHE "INVERSE":

La funzione seno:

$$\sin : \mathbb{R} \longrightarrow \mathbb{R}$$

NON è invertibile, poiché non è né su né 1-1.

Tuttavia se restringiamo il suo codominio:

$$\sin : \mathbb{R} \longrightarrow [-1, 1]$$

è SURIETTIVA

(ma non INIETTIVA)

Se però sostituiamo alla funzione seno la sua restrizione all'interv. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$:

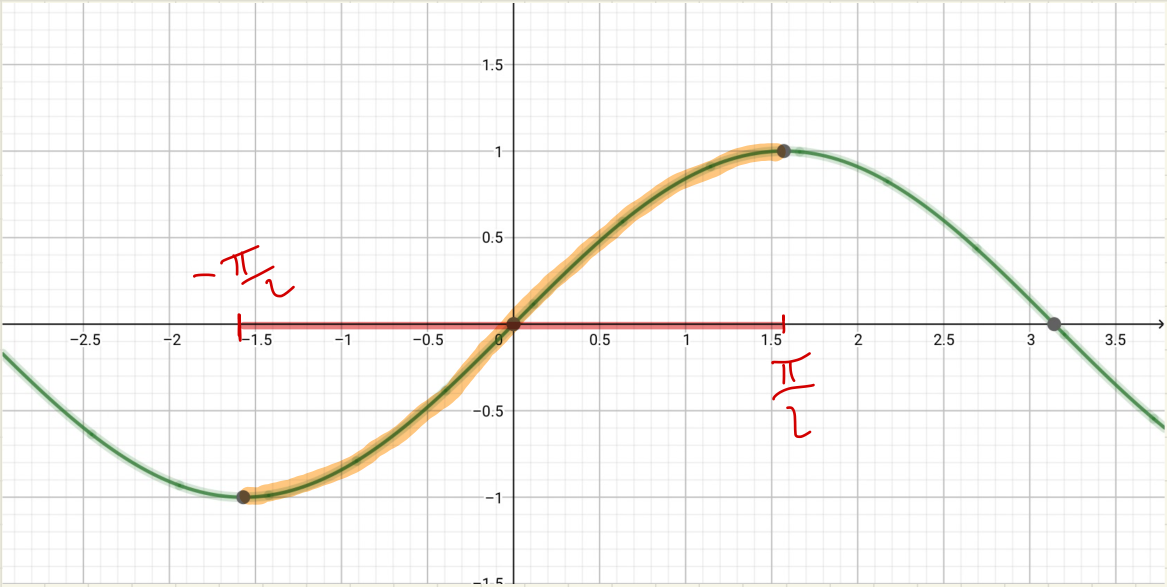
$$\sin\Big|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

$\Longrightarrow$ $\sin\Big|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$ è su e 1-1

$\Longrightarrow$ è invertibile

$$\sin\Big|_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

funzione inversa


1.5
1
0.5
−0.5
−1
−1.5
−2.5
−2
−1.5
−1
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
$-\pi/2$
$\pi/2$

$$\left(\sin\big|_{\left[-\frac{\pi}{2},\frac{\pi}{2}\right]}\right)^{-1}(y) =: \arcsin y$$

arcoseno di y

$$\arcsin : [-1,1] \longrightarrow \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$y \longmapsto \arcsin y$$

ATTENZIONE:

Il fatto che arcsin sia l'inversa di una restrizione del seno ha delle conseguenze:

$\forall y \in [-1, 1]$:

$$\sin(\arcsin y) = y$$

$\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (e non in $\mathbb{R}$!)

$$\arcsin(\sin x) = x$$

(<u>Esempio</u>:

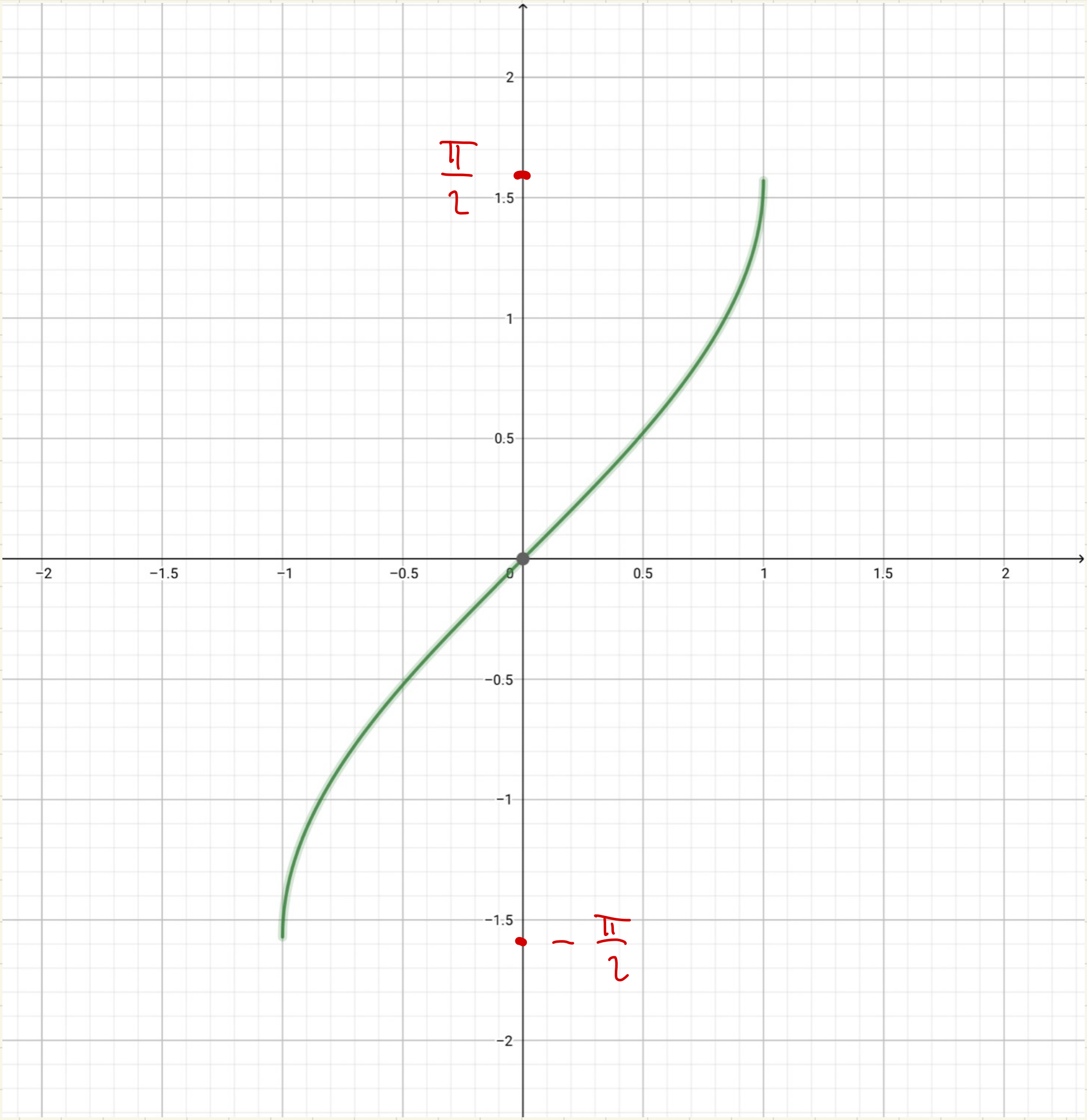
$x = \pi$ (nota che $\pi \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$)

$$\arcsin(\sin \pi) =$$

$$= \arcsin(0) = 0 \neq \pi$$

)

$y = \arcsin x$



Anche la funzione coseno non è invertibile; consideriamo la sua restrizione a $[0, \pi]$:

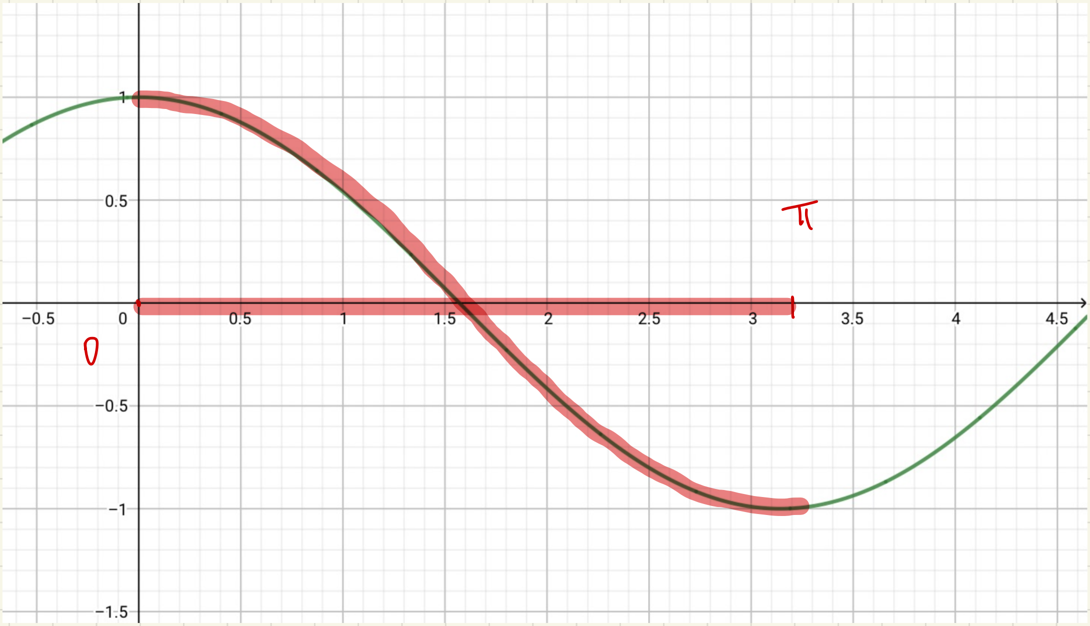
$$\cos\big|_{[0,\pi]} : [0, \pi] \longrightarrow [-1, 1]$$

$\Longrightarrow$ $\cos\big|_{[0,\pi]}$ è su e 1-1

$\Longrightarrow$ è invertibile

$$\cos\big|_{[0,\pi]} : [0, \pi] \longrightarrow [-1, 1]$$

funzione inversa



1
0.5
−0.5
−1
−1.5
−0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
π
0

$$\left(\cos\big|_{[0,\pi]}\right)^{-1}(y) =: \arccos y$$

arcocoseno di y

$$\arccos : [-1,1] \longrightarrow [0,\pi]$$

$$y \longmapsto \arccos y$$

ATTENZIONE:

Il fatto che arccos sia l'inversa di una restrizione del coseno ha delle conseguenze come prima.

$\forall y \in [-1, 1]$:

$$\cos(\arccos y) = y$$

$\forall x \in [0, \pi]$ (e non in $\mathbb{R}$!)

$$\arccos(\cos x) = x$$

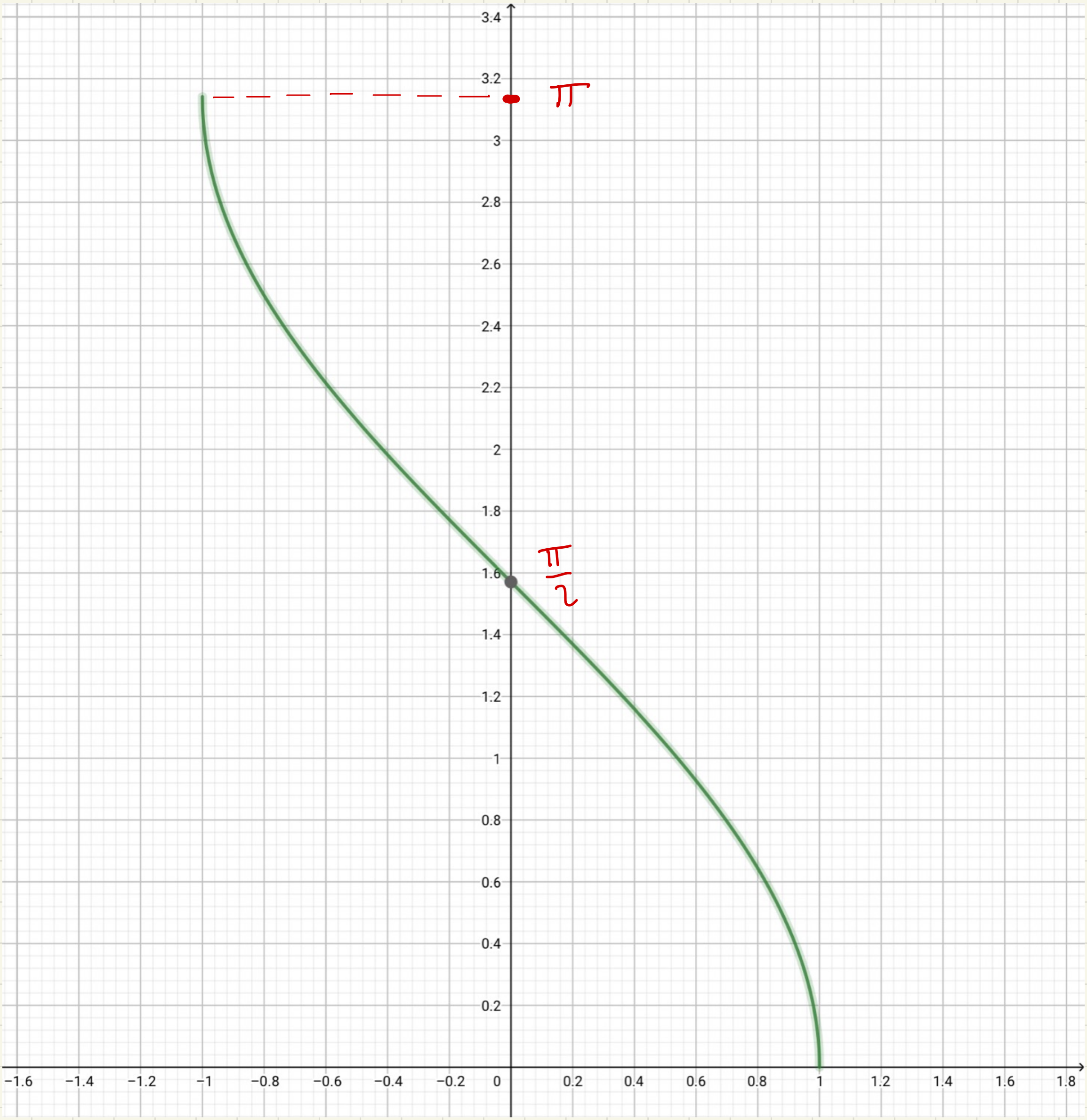
(Esempio:

$x = \frac{3\pi}{2}$ (nota che $\frac{3\pi}{2} \notin [0, \pi]$)

$$\arccos\left(\cos \frac{3\pi}{2}\right) =$$

$$= \arccos(0) = \frac{\pi}{2} \neq \frac{3\pi}{2}$$)

$y = \arccos x$



3.4
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
−1.6
−1.4
−1.2
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
π
$\frac{\pi}{2}$

Anche la funzione tangente non è invertibile (non è biunivoca) ma lo è la sua restrizione:

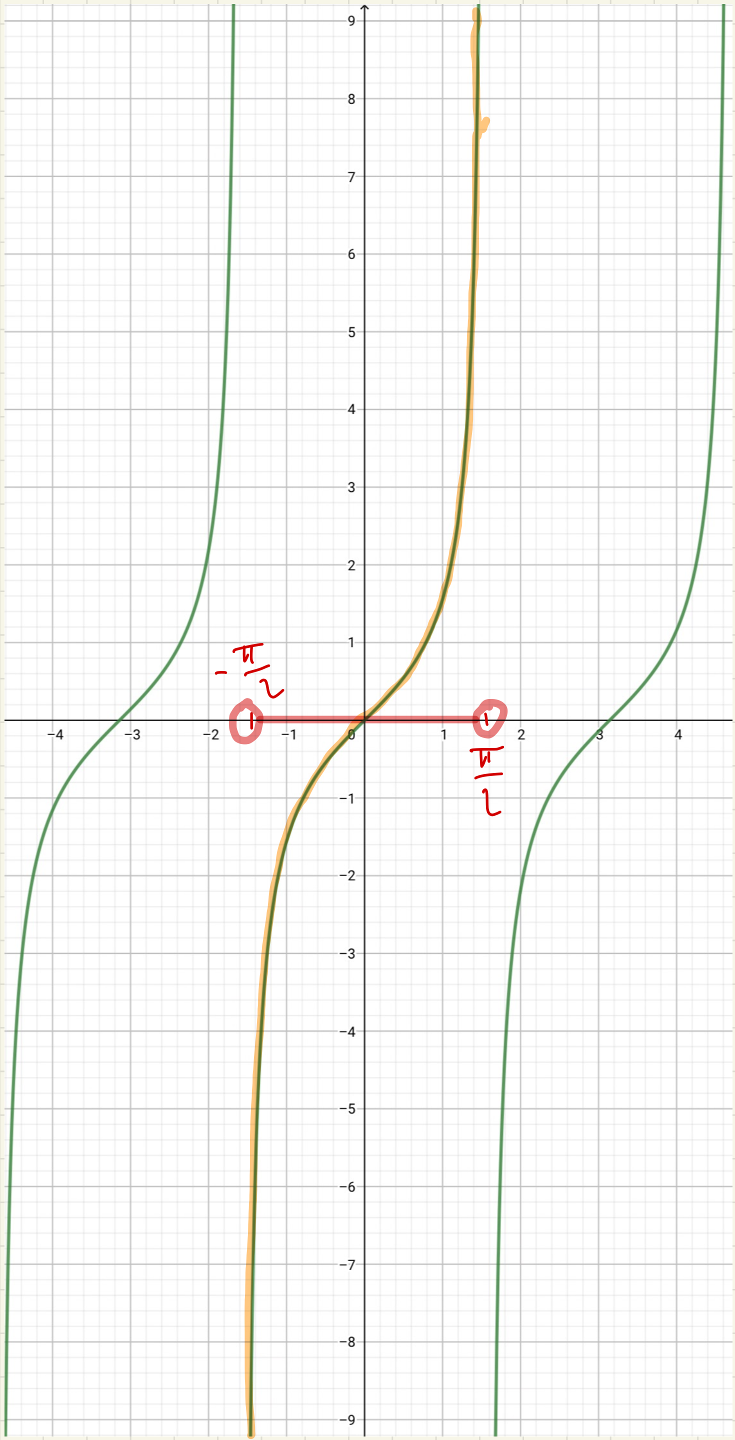
$$tg\big|_{]-\frac{\pi}{2},\frac{\pi}{2}[} : \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\longrightarrow \mathbb{R}$$

$\Longrightarrow$ $tg\big|_{]-\frac{\pi}{2},\frac{\pi}{2}[}$ è su e 1-1

$\Longrightarrow$ è invertibile

$$tg\big|_{]-\frac{\pi}{2},\frac{\pi}{2}[} : \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\longrightarrow \mathbb{R}$$

funzione inversa



9
8
7
6
5
4
3
2
1
−4
−3
−2
−1
0
1
2
3
4
−1
−2
−3
−4
−5
−6
−7
−8
−9
$-\frac{\pi}{2}$
$\frac{\pi}{2}$

$(\mathrm{tg}|_{]-\frac{\pi}{2},\frac{\pi}{2}[})^{-1}(y) =: \mathrm{arctg}\, y$

(arctan y)

arcotangente di y

$$\mathrm{arctg} : \mathbb{R} \longrightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

$$y \longmapsto \mathrm{arctg}\, y$$

ATTENZIONE:

Il fatto che arctg sia l'inversa di una restrizione della tangente ha delle conseguenze!

$\forall y \in \mathbb{R}$:

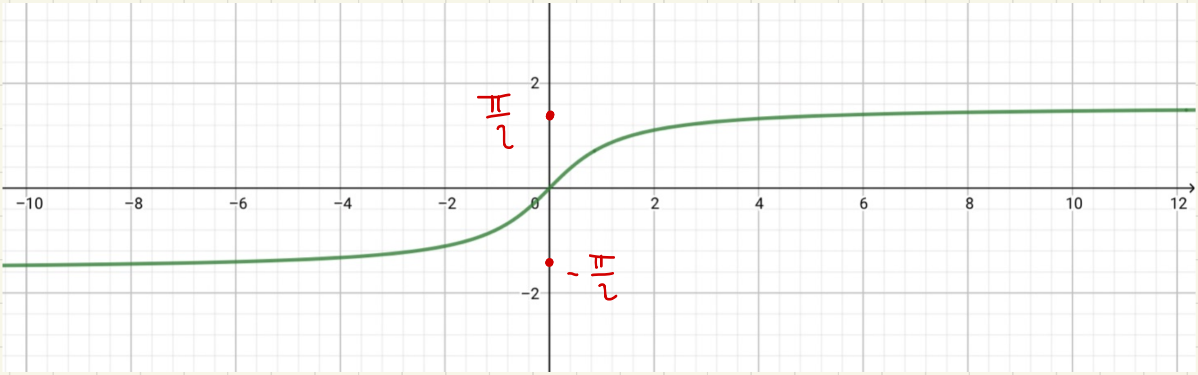
$$\operatorname{tg}(\operatorname{arctg} y) = y$$

$\forall x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ (e non in $\mathbb{R}$!)

$$\operatorname{arctg}(\operatorname{tg} x) = x$$

Grafico di $y = \operatorname{arctg} x$

$(= \arctan x)$



INTRODUZIONE ALLA NOZIONE DI LIMITE:

DEF.: (Intorno sferico di un punto $x_0 \in \mathbb{R}$ di raggio r)

$x_0 \in \mathbb{R}$, $\quad r \in \mathbb{R} : r > 0$

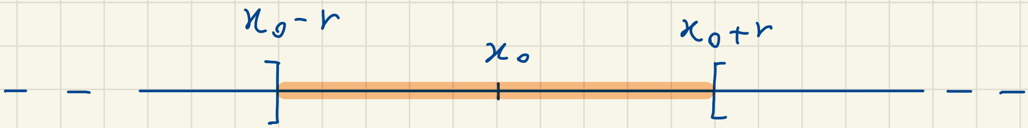
Si dice intorno (sferico) di centro x_0 e raggio r:

$$I_r(x_0) = \{ x \in \mathbb{R} \mid |x - x_0| < r \}$$

$$\left(\shortparallel \; B_r(x_0) \right)$$

$$\Updownarrow$$

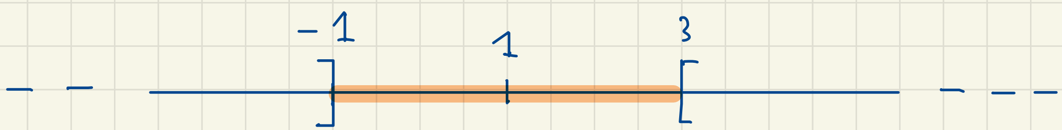
$$x_0 - r < x < x_0 + r$$



$$I_r(x_0) = \left] x_0 - r, x_0 + r \right[$$

Esempio:

$$I_2(1) = \{ x \in \mathbb{R} \mid |x-1| < 2 \}$$
$$= \{ x \in \mathbb{R} \mid -1 < x < 3 \}$$
$$= \,]-1, 3[$$



Nota: Un intervallo della forma $]a,b[$, $]-\infty,a[$, $]b,+\infty[$ si dice intervallo aperto.

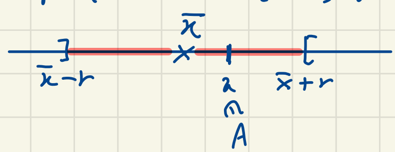
DEF. : (Punto di accumulazione di un insieme A)

$A \subseteq \mathbb{R}$

$\bar{x} \in \mathbb{R}$ si dice **punto di accumulazione** di A se:

$\forall r > 0$:

$$A \cap \left(I_r(\bar{x}) \setminus \{\bar{x}\} \right) \neq \emptyset$$

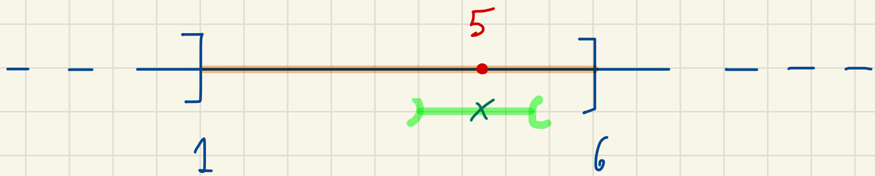


$$D(A) = \left\{ \bar{x} \in \mathbb{R} \;\middle|\; \bar{x} \text{ è di accumulazione per } A \right\}$$

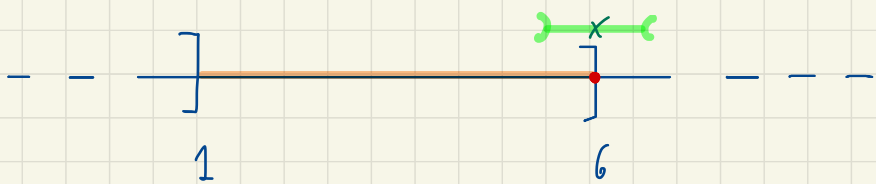
Idea: $\bar{x}$ si dice punto di accumulazione di A se ci si può avvicinare arbitrariamente a $\bar{x}$, rimanendo in A!

Esempi:

- 5 è di accumulazione per $A =]1, 6]$



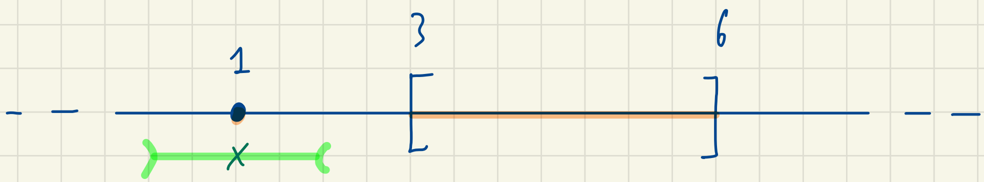
- 6 è di accumulazione per $A =]1, 6]$



- 1 è di accumulazione per $A =]1, 6]$

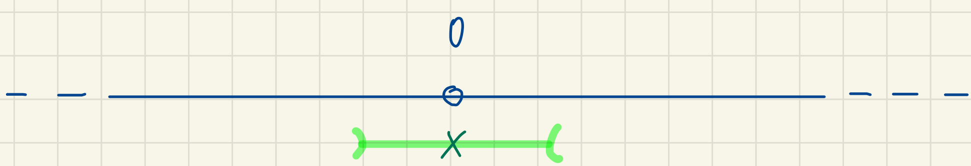
- $A = \{1\} \cup [3, 6]$

1 NON è di accumulazione per A.



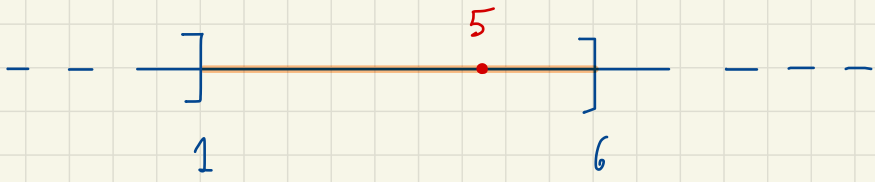
- $A = \mathbb{R} \setminus \{0\}$ $\qquad (0 \notin A)$

0 è di accumulazione per A



Esempi:

- 5 è di accumulazione per $A = \,]1, 6]$



$D(A) = [1, 6]$ (Nota: $A \subsetneq D(A)$)

- 6 è di accumulazione per $A = [1, 6] \cup \{7\}$

$D(A) = [1, 6]$ (Nota: $D(A) \subsetneq A$)

PROPOSIZIONE:

$A \subseteq \mathbb{R}$, $\bar{x} \in \mathbb{R}$

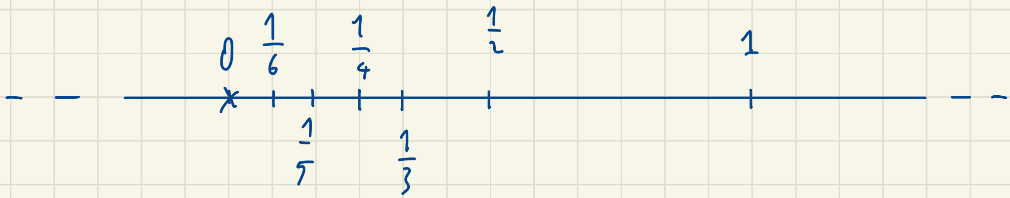
$\bar{x}$ è di accumulazione per A

se e solo se

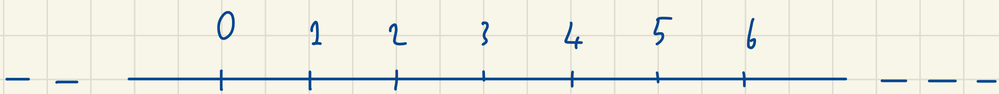
$\exists (a_n)_n \subseteq A$ t.c.:

1. $a_n \neq \bar{x} \quad \forall n$
2. $a_n \xrightarrow[n \to +\infty]{} \bar{x}$

- $A = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$

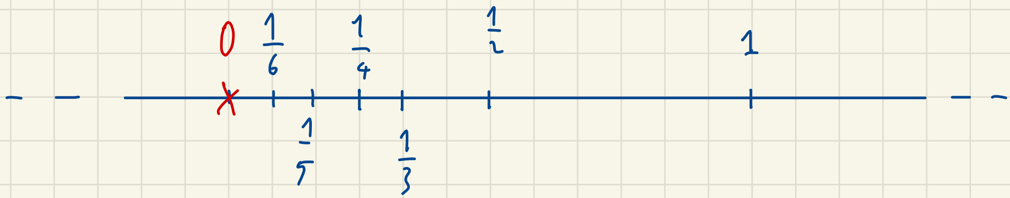


$D(A) = ?$ $\qquad \left(\lim_{n \to +\infty} \frac{1}{n} = 0 \right)$

- 

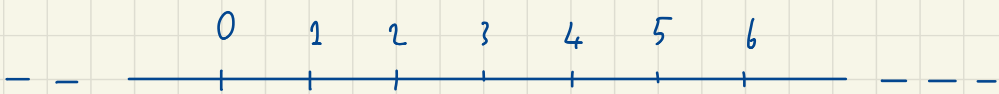
$D(\mathbb{N}) = ?$

- $A = \left\{ \frac{1}{n} \;\middle|\; n \in \mathbb{N}^* \right\}$



$$D(A) = \{0\}$$

-



$$D(\mathbb{N}) = \emptyset$$

Prima di introdurre la nozione di limite $\lim_{x \to x_0} f(x) = l$, discutiamo un esempio.

$$f : D(f) \longrightarrow \mathbb{R}$$

$$f(x) = \frac{x^3 - 4x}{x - 2}$$

$$D(f) = \mathbb{R} \setminus \{2\}$$

2 è un punto di accumulazione di $D(f)$.

x	$f(x)$
$2+1=3$	15
$2+\frac{1}{10}=2{,}1$	$8{,}61$
$2+\frac{1}{100}=2{,}01$	$8{,}0601$
$2+\frac{1}{1000}=2{,}001$	$8{,}006001$
$2+\frac{1}{10.000}=2{,}0001$	$8{,}00006$

x	$f(x)$
$2 - 1 = 1$	3
$2 - \frac{1}{10} = 1{,}9$	$7{,}41$
$2 - \frac{1}{100} = 1{,}99$	$7{,}9401$
$2 - \frac{1}{1000} = 1{,}999$	$7{,}994001$
$2 - \frac{1}{10'000} = 1{,}9999$	$7{,}99940001$
$2 - \frac{1}{100'000} = 1{,}99999$	$7{,}99994 \ldots$

" sembra " $f(x) \xrightarrow[x \to 2]{} \ell$

<u>ATTENZIONE</u> : $2 \notin D(f)$

DEF. (limite finito al finito)

$f : A \longrightarrow \mathbb{R}$

$x_0 \in D(A)$, $\ell \in \mathbb{R}$

Si dice che $\lim_{x \to x_0} f(x) = \ell$ se

$\forall \varepsilon > 0, \exists \delta = \delta(x_0, \varepsilon) > 0 : \forall x \in A : 0 < |x - x_0| < \delta$

$\Longrightarrow |f(x) - \ell| < \varepsilon$

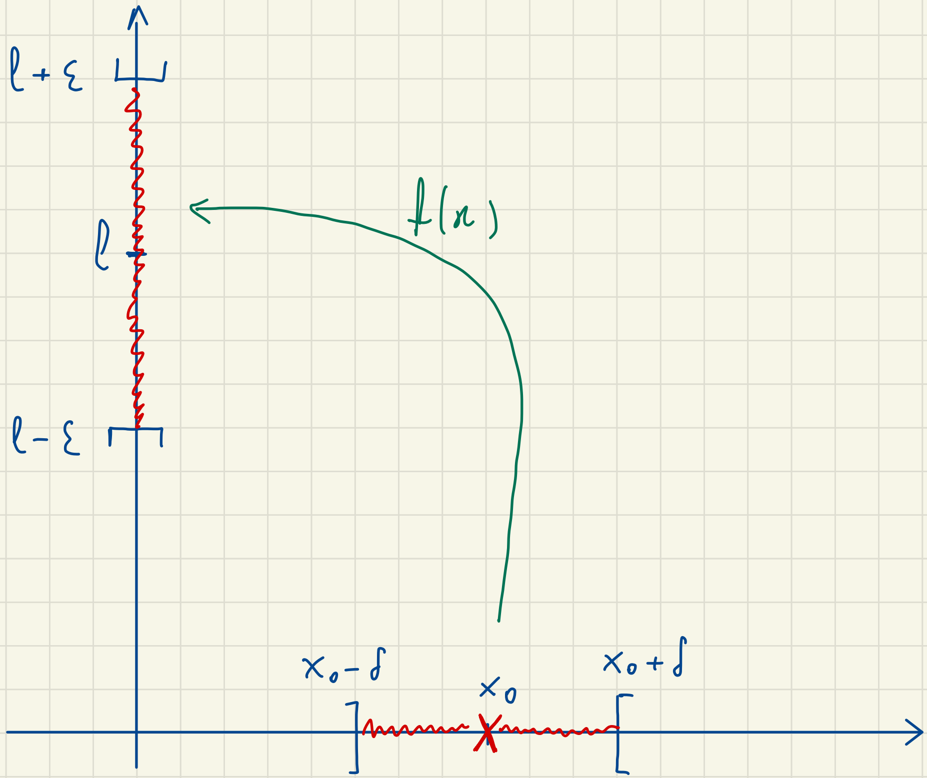
$x \neq x_0$

$x_0 - \delta < x < x_0 + \delta$

$\ell - \varepsilon < f(x) < \ell + \varepsilon$

$$\lim_{x \to x_0} f(x) = \ell$$

$$\forall \varepsilon > 0, \quad \exists \delta > 0 :$$



Esempio:

$$\lim_{x \to 0} x^2 = 0$$

$$\forall \varepsilon > 0, \quad \exists \delta = \delta(0, \varepsilon) > 0 : \forall x \in \mathbb{R} : 0 < |x - 0| < \delta$$

$$\Longrightarrow \quad |x^2 - 0| < \varepsilon$$

(DA PROVARE)

Esempio:

$$x \neq 0,\ -\delta < x < \delta$$
$$\Updownarrow$$
$$0 < |x| < \delta$$
$$\Updownarrow$$

$$\lim_{x \to 0} x^2 = 0$$

$$\forall \varepsilon > 0,\ \exists \delta = \delta(0, \varepsilon) > 0 : \forall x \in \mathbb{R} : 0 < |x - 0| < \delta$$

$$\Longrightarrow |x^2 - 0| < \varepsilon$$

$$|x^2 - 0| = |x^2| = x^2$$

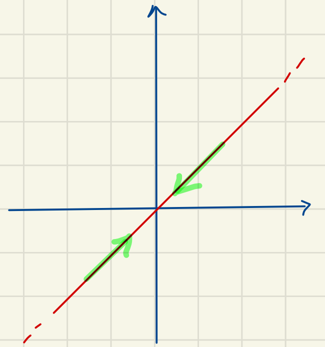
$$x^2 < \varepsilon \iff -\underbrace{\sqrt{\varepsilon}}_{=:\ \delta} < x < \sqrt{\varepsilon}$$

Se scegliamo $\delta = \sqrt{\varepsilon}$ allora:

$$-\sqrt{\varepsilon} < x < \sqrt{\varepsilon} \Longrightarrow x^2 < \varepsilon$$

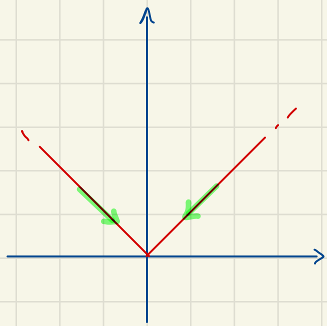
Esercizio:

$$\lim_{x \to 0} x = 0$$

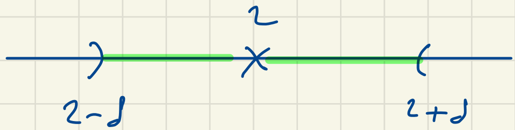


$$\left(\text{scegliere } \delta := \varepsilon \right)$$

$$\lim_{x \to 0} |x| = 0$$



Esempio:



$x \neq +2$

$$2-\delta < x < 2+\delta$$

$$\Updownarrow$$

$$\lim_{x \to 2} x^2 = 4$$

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \forall x \in \mathbb{R} : 0 < |x-2| < \delta$$

$$\Rightarrow \quad |x^2 - 4| < \varepsilon$$

$$\Updownarrow$$

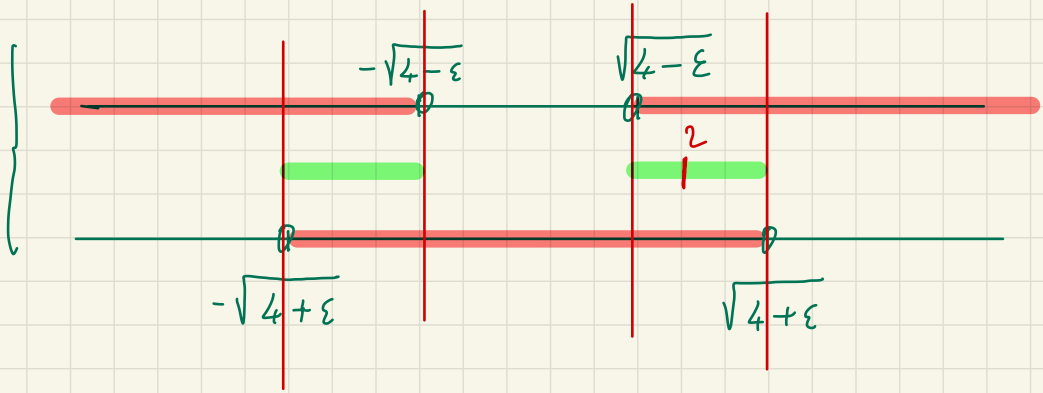
$$-\varepsilon < x^2 - 4 < \varepsilon$$

$$\Updownarrow$$

$$4 - \varepsilon < x^2 < 4 + \varepsilon$$

$$\begin{cases} x^2 > 4 - \varepsilon \\ x^2 < 4 + \varepsilon \end{cases}$$

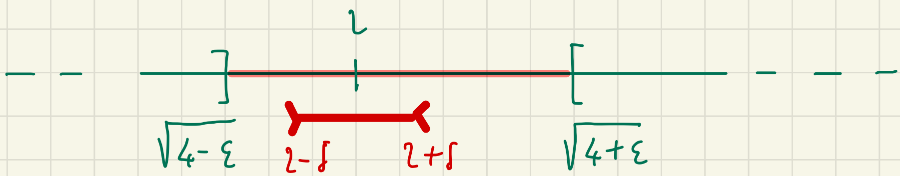
$$\begin{cases} x^2 > 4 - \varepsilon \\ x^2 < 4 + \varepsilon \end{cases} \qquad \begin{cases} x < -\sqrt{4-\varepsilon} \quad \vee \quad x > \sqrt{4-\varepsilon} \\ -\sqrt{4+\varepsilon} < x < \sqrt{4+\varepsilon} \end{cases}$$



Quindi $|x^2 - 4| < \varepsilon$ se e solo se:

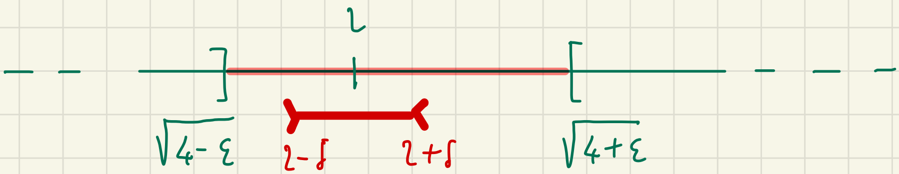
$$-\sqrt{4+\varepsilon} < x < -\sqrt{4-\varepsilon} \quad \vee \quad \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$$

contiene 2



Se scegliamo: $\sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$

contiene 2



$$\Downarrow$$

$$|x^2 - 4| < \varepsilon$$

Si dimostrerà in seguito

che se

$$p(x) = \sum_{j=0}^{n} a_j \cdot x^j$$

(polinomio di grado n)

allora:

$$\lim_{x \to \bar{x}} p(x) = p(\bar{x})$$

Es.:

$$\lim_{x \to 2} (3x^2 - x + 1) = 11$$

$$\lim_{x \to -2} x^3 = (-2)^3 = -8$$

Oss.:

$$\lim_{x\to\bar{x}} f(x) = 0 \iff \lim_{x\to\bar{x}} |f(x)| = 0$$

$\forall \varepsilon>0,\ \exists \delta>0 : \forall x\in D(f):$

$$0<|x-\bar{x}|<\delta \implies \underbrace{|f(x)-0|}_{=|f(x)|}<\varepsilon$$

$\forall \varepsilon>0 : \exists \delta>0 : \forall x\in D(|f|)=D(f)$

$$0<|x-\bar{x}|<\delta \implies \big|\,|f(x)|-0\,\big|<\varepsilon$$

$$\big|\,|f(x)|-0\,\big| = \big|\,|f(x)|\,\big| = |f(x)|$$

Sono uguali!!

$$\lim_{x \to -2} x^3 = (-2)^3 = -8$$

$$\lim_{x \to -2} |x^3| = |(-2)^3| = 8$$

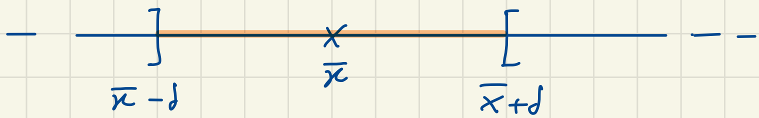
TEOREMA (di permanenza del segno):

$f : A \longrightarrow \mathbb{R}$, $\bar{x} \in D(A)$

$\lim\limits_{x \to \bar{x}} f(x) = \ell \in \mathbb{R}$, $\ell > 0$ $(\ell < 0)$

Allora:

$\exists \delta > 0 : \forall x \in A, \ \bar{x} - \delta < x < \bar{x} + \delta, \ x \neq \bar{x}$



$\Longrightarrow \quad f(x) > 0 \quad (f(x) < 0)$

DIM.:

Esercizio (scegliere $\varepsilon = \frac{|\ell|}{2}$)

TEOREMA (del confronto):

$$f, g, h : A \longrightarrow \mathbb{R}$$

$$x_0 \in D(A)$$

Supponiamo che:

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = \ell \in \mathbb{R}$$

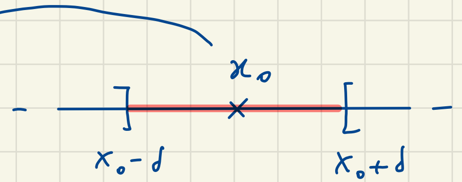
$\exists \, \delta > 0 :$

$$g(x) \le f(x) \le h(x) \, , \quad \forall x \in [A \cap I_\delta(x_0)] \setminus \{x_0\}$$

Allora:

$$\lim_{x \to x_0} f(x) = \ell$$

$$
\begin{tikzcd}
g(x) \arrow[d, "x \to x_0"] & \le f(x) \le & h(x) \arrow[d, "x \to x_0"] \\
\ell & & \ell
\end{tikzcd}
$$



$$\Longrightarrow \quad f(x) \longrightarrow \ell$$

Per $\lim_{x \to \bar{x}} f(x)$ vale l'algebra dei limiti più vista per le successioni:

- $\lim_{x \to \bar{x}} \left(f(x) \pm g(x) \right) = \lim_{x \to \bar{x}} f(x) \pm \lim_{x \to \bar{x}} g(x)$
- $\lim_{x \to \bar{x}} \left(f(x) \cdot g(x) \right) = \lim_{x \to \bar{x}} f(x) \cdot \lim_{x \to \bar{x}} g(x)$
- $\lim_{x \to \bar{x}} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$

$\left(g(x) \neq 0 \text{ se } x \sim \bar{x} \right)$

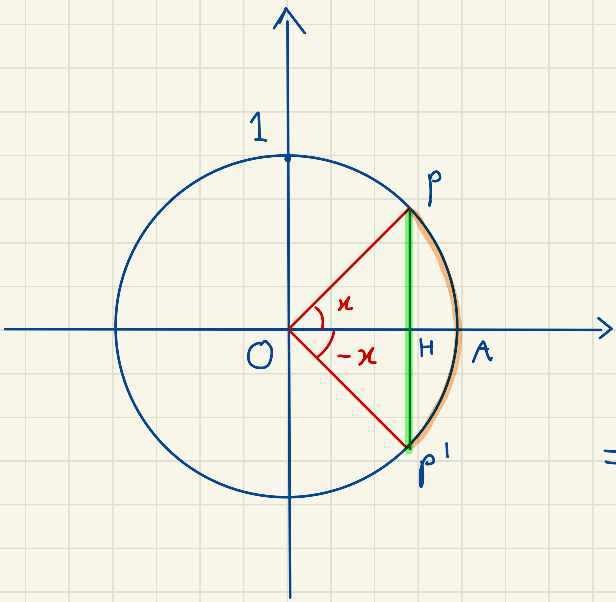
$\left(\lim_{x \to \bar{x}} g(x) \neq 0 \right)$

PROP.:

$$\lim_{x \to 0} \sin x = 0 \quad (= \sin 0)$$

$$\lim_{x \to 0} \cos x = 1 \quad (= \cos 0)$$

DIM.:



$$\begin{array}{ccc} |\overline{PP'}| & < & |\overset{\frown}{PP'}| \\ \parallel & & \parallel \\ 2|\overline{PH}| & & 2|\overset{\frown}{PA}| \end{array}$$

$$\Longrightarrow \quad |\overline{PH}| < |\overset{\frown}{PA}|$$

$$0 \leq |\sin x| < |x|$$

$$\begin{tikzcd}[column sep=small]
0 \arrow[d, "x \to 0"] & \le & |\sin x| & < & |x| \arrow[d, "x \to 0"] \\
0 & & & & 0
\end{tikzcd}$$

Dal teorema del confronto:

$$\lim_{x \to 0} |\sin x| = 0$$

$$\Downarrow$$

$$\lim_{x \to 0} \sin x = 0$$

Dalla lezione scorsa

$$\cos t = \cos\left(2 \cdot \left(\frac{t}{2}\right)\right) = 1 - 2\sin^2\left(\frac{t}{2}\right)$$

$$\Longrightarrow \quad 1 - \cos t = 2 \sin^2\left(\frac{t}{2}\right)$$

$$\left|\sin\left(\frac{t}{2}\right)\right| < \left|\frac{t}{2}\right|$$

$$\Downarrow$$

$$\sin^2\left(\frac{t}{2}\right) < \frac{t^2}{4}$$

$$2\sin^2\left(\frac{t}{2}\right) < 2\cdot\frac{t^2}{4} = \frac{t^2}{2}$$

$$\underset{\displaystyle \downarrow \atop \displaystyle 0}{0} \leq 1-\cos t = 2\sin^2\left(\frac{t}{2}\right) < \underset{\displaystyle \big\downarrow {\scriptstyle t\to 0} \atop \displaystyle 0}{\frac{t^2}{2}}$$

Dal teorema del confronto:

$$\lim_{t\to 0}(1-\cos t) = 0$$

$$\Longrightarrow \lim_{t\to 0}\cos t = 1$$

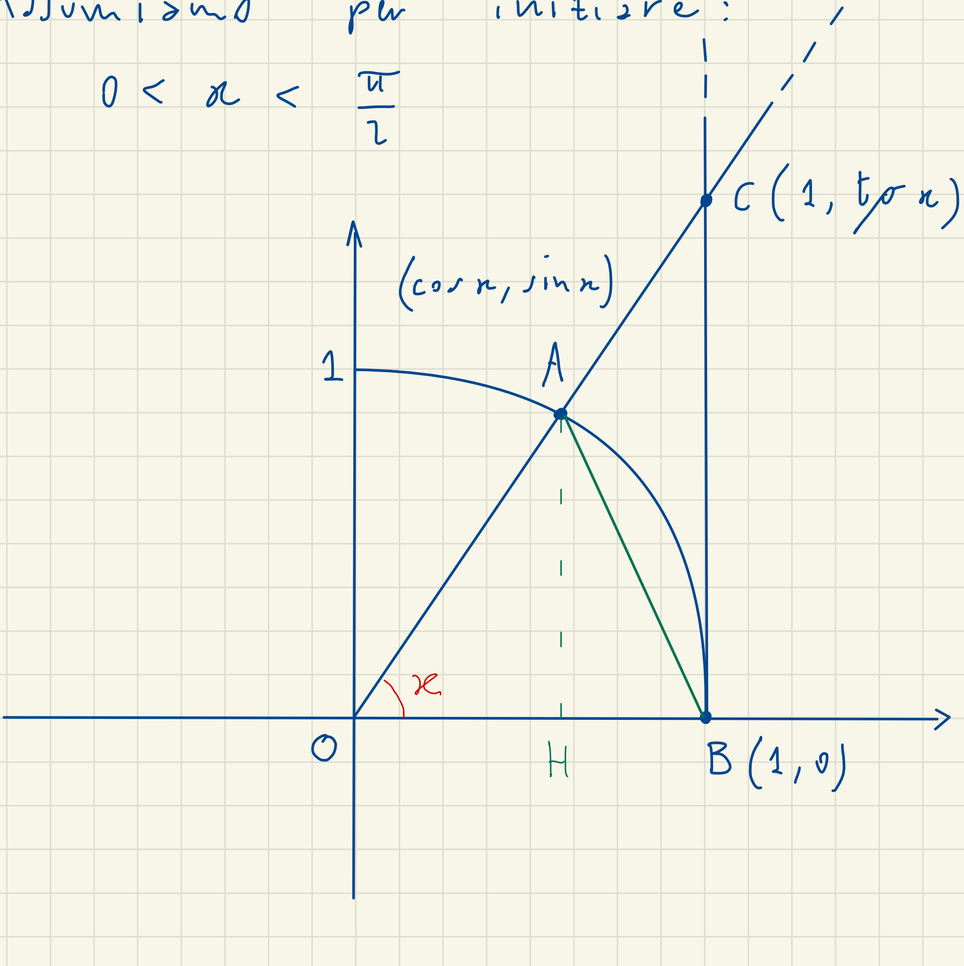
TEOREMA:

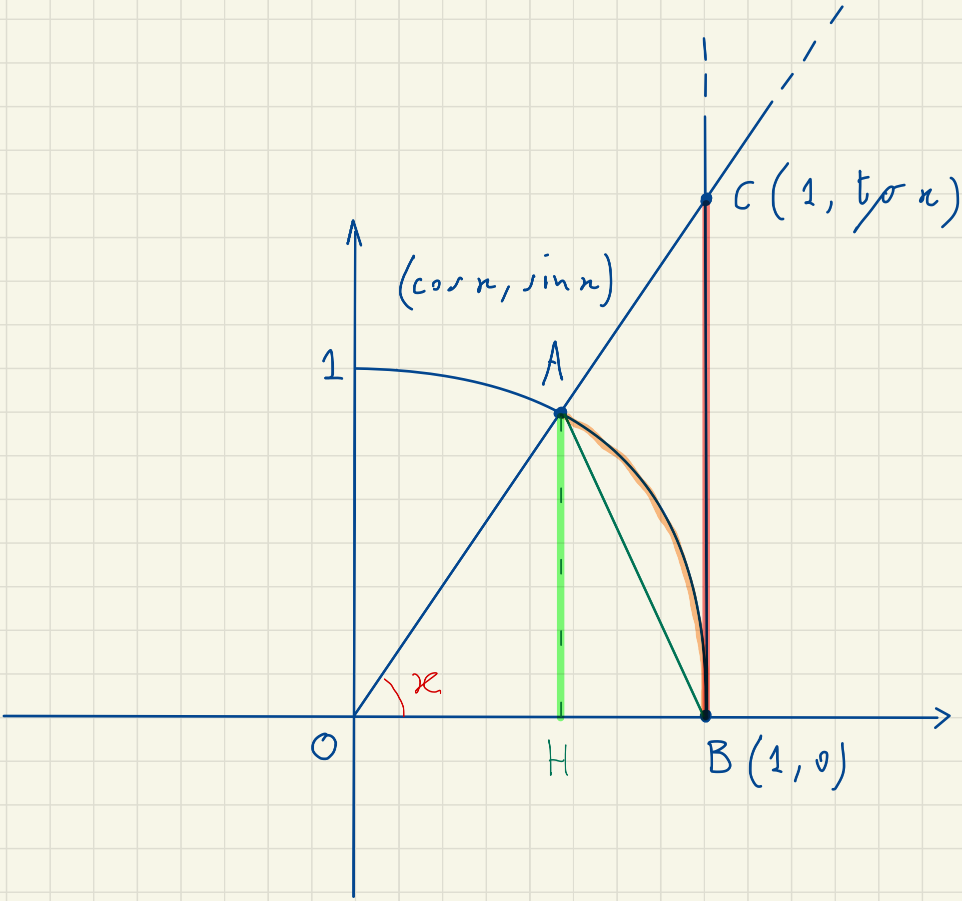
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

DIM.:

Assumiamo per iniziare:

$$0 < x < \frac{\pi}{2}$$





Si mostra che

$$\overline{AH} \leq |\overset{\frown}{AB}| \leq \overline{BC}$$

"già provato"

da provare via la geometria euclidea

$$\sin x \leq x \leq \operatorname{tg} x$$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \qquad \left(0 < x < \frac{\pi}{2}\right)$$

> 0 $\quad$ > 0

Passando ai reciproci:

$$1 \geq \frac{\sin x}{x} \geq \cos x \qquad \left(0 < x < \frac{\pi}{2}\right)$$

Siccome $\sin x$ è una funzione dispari e $\cos x$ è pari:

$$\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = \frac{\sin x}{x}$$

$$\cos(-x) = \cos x$$

Quindi la relazione sopra vale anche per x negativo:

$$-\frac{\pi}{2} < x < 0$$

$$1 \geq \frac{\sin x}{x} \geq \cos x \qquad \left(0 < |x| < \frac{\pi}{2}\right)$$

Passando ai reciproci:

$$\begin{array}{ccccc} 1 & \geq & \dfrac{\sin x}{x} & \geq & \cos x \\ \downarrow & & & & \downarrow {\scriptstyle x \to 0} \\ 1 & & & & 1 \end{array}$$

Dal teorema del confronto:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Dal limite $\lim_{x \to 0} \frac{\sin x}{x} = 1$

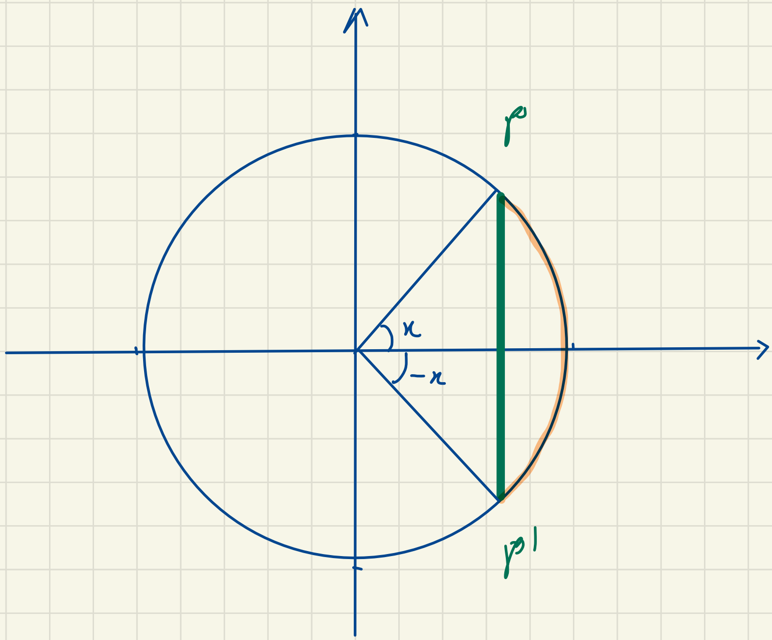
seguirà che le funzioni

goniometriche dirette e inverse

sono continue e derivabili!

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

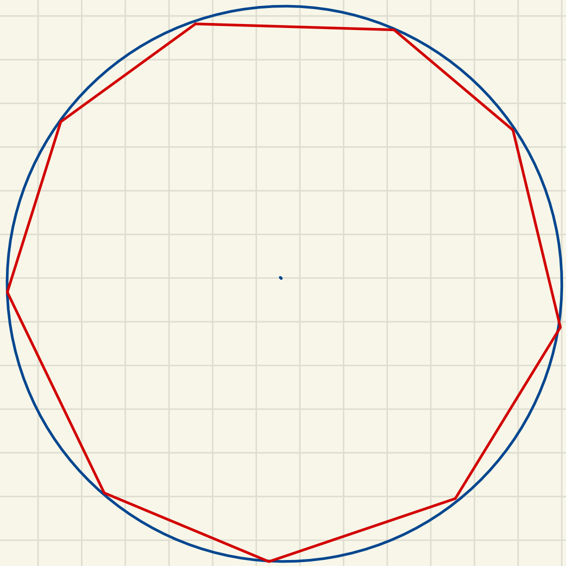
Interpretazione geometrica:



$$\overline{PP'} = 2\sin x \qquad |\widehat{PP'}| = 2x$$

$x \to 0$ $\quad 2\sin x \sim 2x$

PROBLEMA: Come si calcola l'area del cerchio?

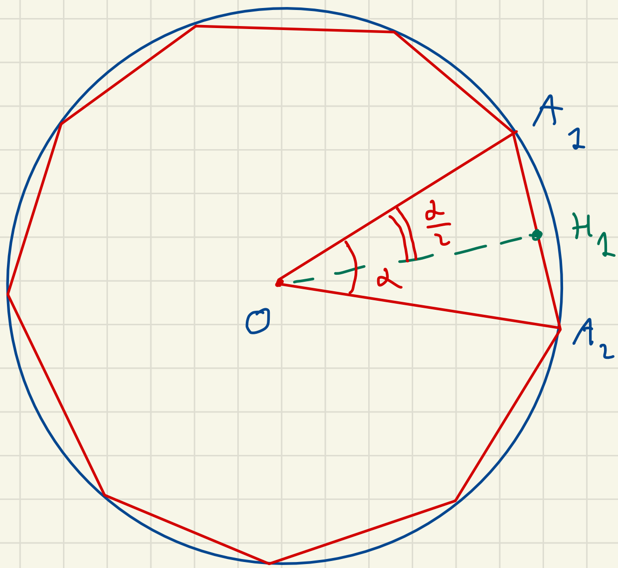


$\mathcal{C}$ cerchio di raggio r

S_n poligono regolare inscritto di n lati inscritto nel cerchio

$$\mathcal{A}(\mathcal{C}) = \lim_{n \to +\infty} \mathcal{A}(S_n)$$

area



$$\alpha = \frac{2\pi}{n} \longrightarrow \frac{\alpha}{2} = \frac{\pi}{n}$$

$$\overline{OH_1} = \overline{OA_1} \cdot \cos\frac{\alpha}{2} = r \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\overline{A_1H_1} = \overline{OA_1} \cdot \sin\frac{\alpha}{2} = r \cdot \sin\left(\frac{\pi}{n}\right)$$

$$A(O\overset{\Delta}{A_1}A_2) = 2 \cdot A(O\overset{\Delta}{A_1}H_1) =$$

$$= \overline{A_1H_1} \cdot \overline{OH_1} =$$

$$= r^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\mathcal{A}(S_n) = n \cdot \mathcal{A}(O\hat{A_1}A_2) =$$

$$= n \cdot r^2 \cdot \sin\left(\frac{\pi}{n}\right) \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\lim_{n \to +\infty} \mathcal{A}(S_n) = ?$$

$$\mathcal{A}(S_n) = r^2 \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{1}{n}} \cdot \cos\left(\frac{\pi}{n}\right) =$$

$$= r^2 \cdot \pi \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \cdot \cos\left(\frac{\pi}{n}\right)$$

$$\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \xrightarrow{n \to +\infty} 1 \qquad \cos\left(\frac{\pi}{n}\right) \xrightarrow{n \to +\infty} 1$$

$$n \to +\infty$$
$$\Downarrow$$
$$\frac{\pi}{n} \to 0$$

$$\mathcal{A}(\mathcal{C}) = \lim_{n \to +\infty} \mathcal{A}(S_n) = \pi \cdot r^2$$

Esercizio:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = ?$$

$$\frac{1 - \cos x}{x^2} = \frac{1 - \cos^2 x}{x^2} \cdot \frac{1}{1 + \cos x}$$

$$= \left(\frac{\sin x}{x} \right)^2 \cdot \frac{1}{1 + \cos x}$$

$$\downarrow \qquad \qquad \downarrow$$

$$1 \qquad \qquad \frac{1}{1+1} = \frac{1}{2}$$

Quindi:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Un <u>secondo limite notevole</u> è:

$$\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a$$

(senza dimostrazione)

$(0 < a, \quad a \neq 1)$

(caso particolare: $a = e$)

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \ln e = 1$$

LIMITE INFINITO

AL FINITO :

Esempio:

$$f(x) = \frac{1}{x^2 - 6x + 9}$$

$$D(f) = \mathbb{R} \setminus \{3\}$$

3 è di accumulazione

per $\mathbb{R} \setminus \{3\}$

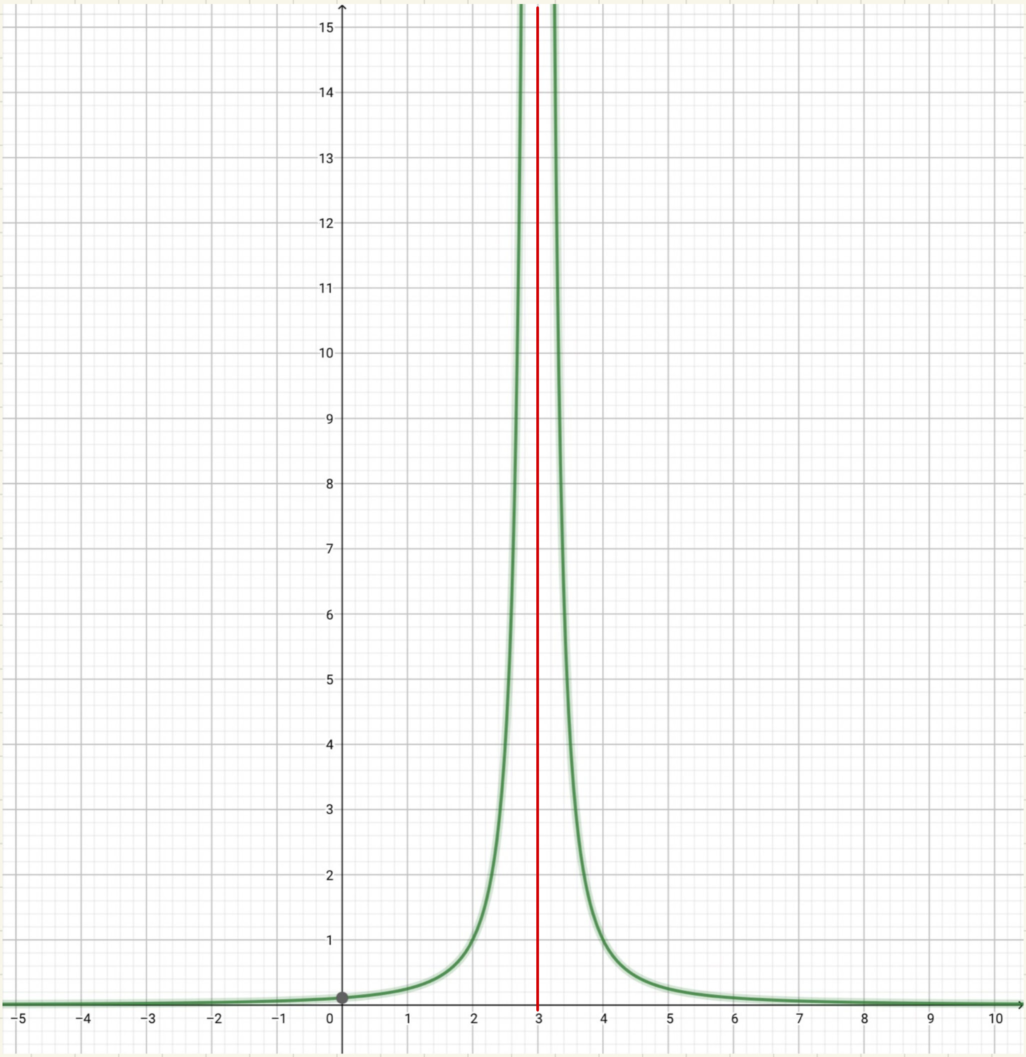
x	$\dfrac{1}{x^2-6x+9}$
$3+\dfrac{1}{10}$	100
$3+\dfrac{1}{100}$	$10'000$
$3+\dfrac{1}{1'000}$	$1'000'000$
$\vdots$	$\vdots$

x	$\dfrac{1}{x^2-6x+9}$
$3-\dfrac{1}{10}$	100
$3-\dfrac{1}{100}$	$10'000$
$3-\dfrac{1}{1'000}$	$1'000'000$

Idea :

$$x \longrightarrow 3 \implies f(x) \longrightarrow +\infty$$

$$f(x) = \frac{1}{x^2 - 6x + 9}$$



DEF.: $f: A \longrightarrow \mathbb{R}$, $x_0 \in D(A)$

Si dice che $\lim_{x \to x_0} f(x) = +\infty \ (-\infty)$

se:

$$\forall M \in \mathbb{R}, \ \exists \delta = \delta(x_0, M) > 0 :$$

$$\forall x \in A : \ 0 < |x - x_0| < \delta$$

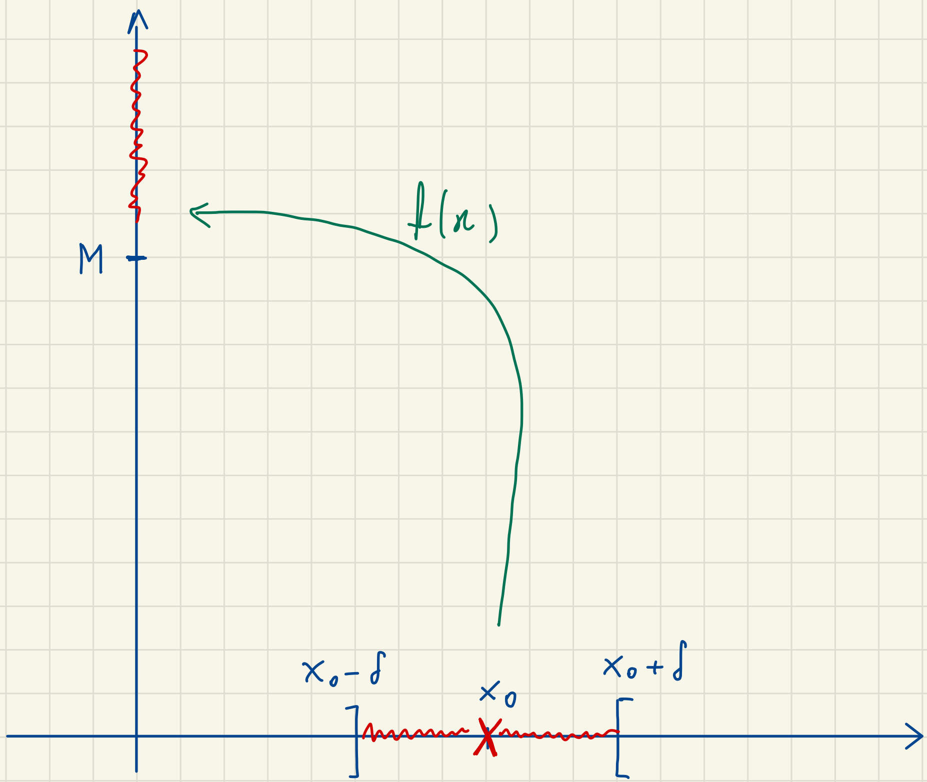
$$\Longrightarrow \quad f(x) > M \quad (f(x) < M)$$

In tal caso la retta $x = x_0$ si

dice ASINTOTO VERTICALE.

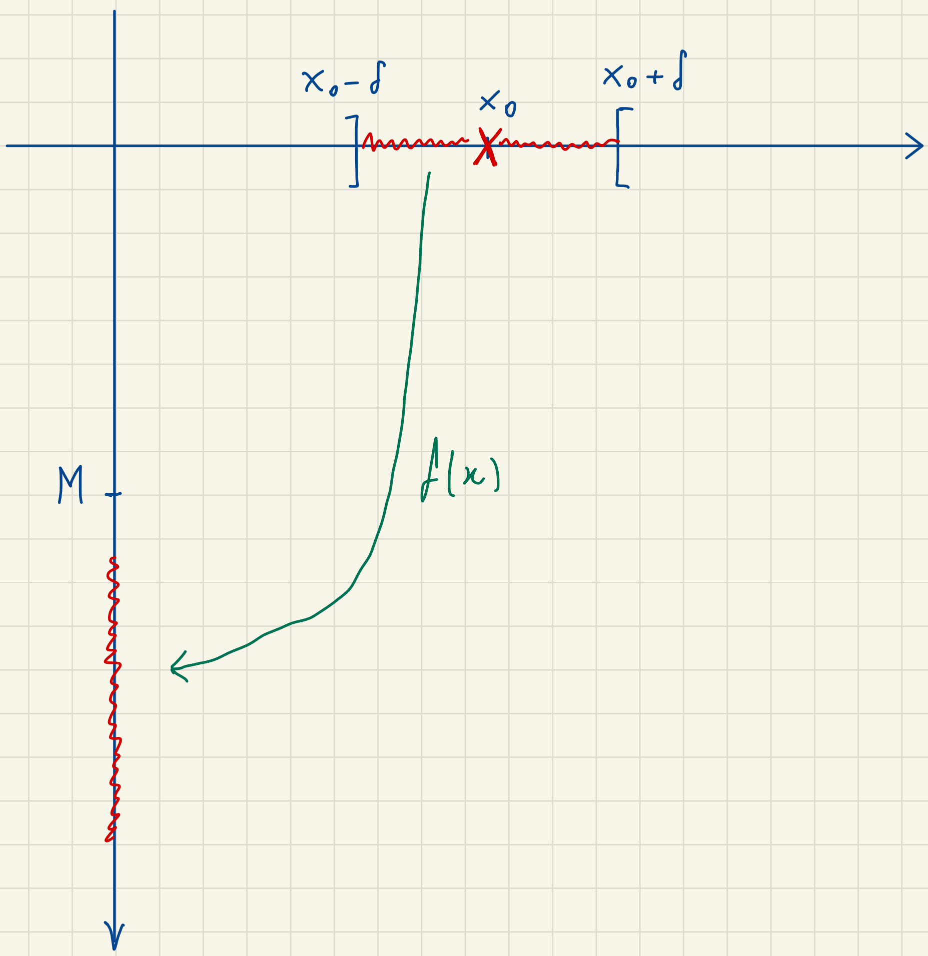
$$\lim_{x \to x_0} f(x) = +\infty$$

$\forall M \in \mathbb{R} : \exists \delta > 0 :$

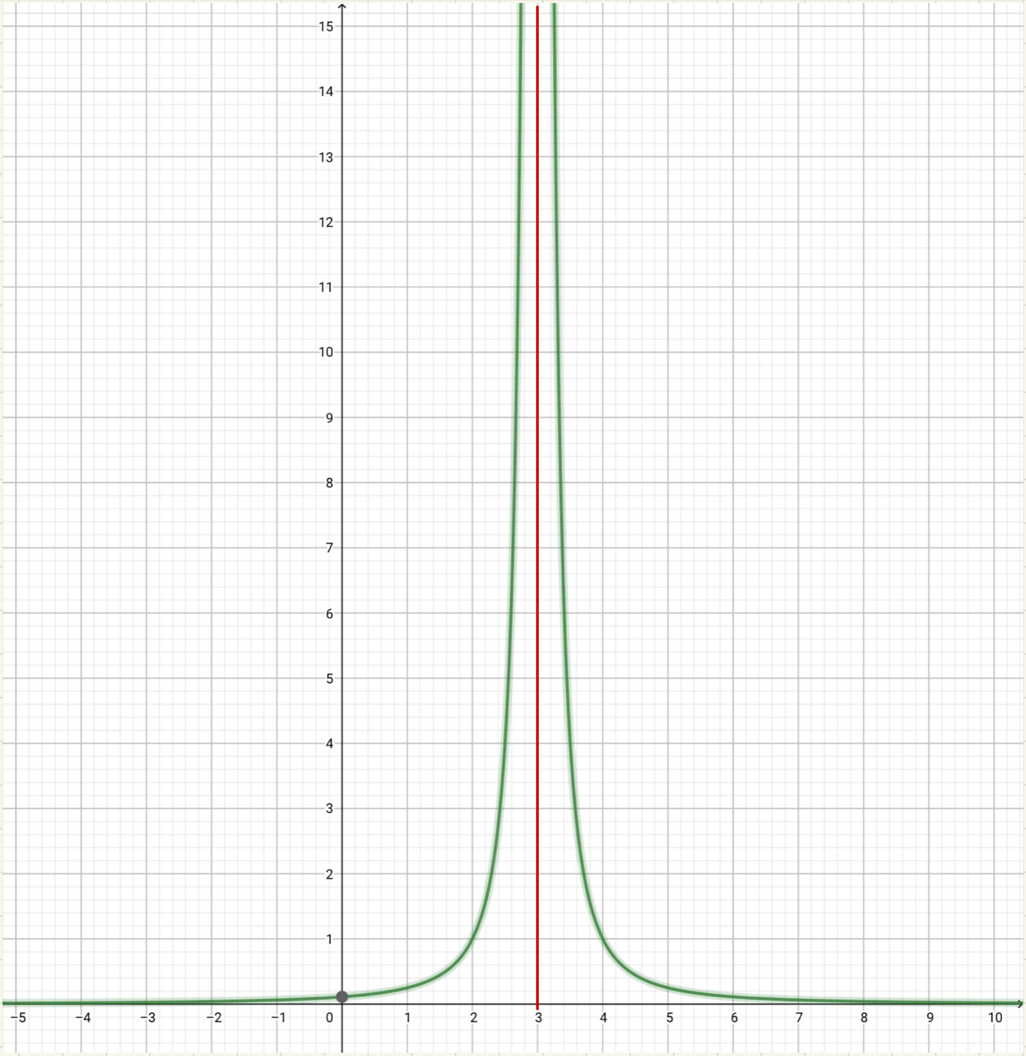


$$\lim_{x \to x_0} f(x) = -\infty$$

$\forall M \in \mathbb{R} : \exists \delta > 0 :$



$$\lim_{x \to 3} \frac{1}{x^2 - 6x + 9} = +\infty$$



$x = 3$ asintoto verticale

Dimostriamo che

$$\lim_{x \to 3} \frac{1}{x^2 - 6x + 9} = +\infty$$

$$x \neq 3$$

$$\frac{1}{x^2 - 6x + 9} > M$$

$$\parallel$$

$$\frac{1}{(x-3)^2} \iff (x-3)^2 < \frac{1}{M}$$

$$\iff -\frac{1}{\sqrt{M}} < x - 3 < \frac{1}{\sqrt{M}} \qquad (x \neq 3)$$

$$\iff 3 - \frac{1}{\sqrt{M}} < x < 3 + \frac{1}{\sqrt{M}} \qquad (x \neq 3)$$

$$\delta = \frac{1}{\sqrt{M}}$$

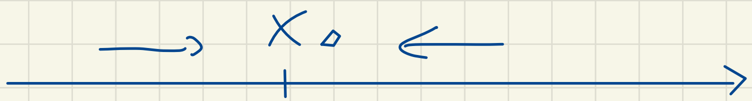
Esercizio:

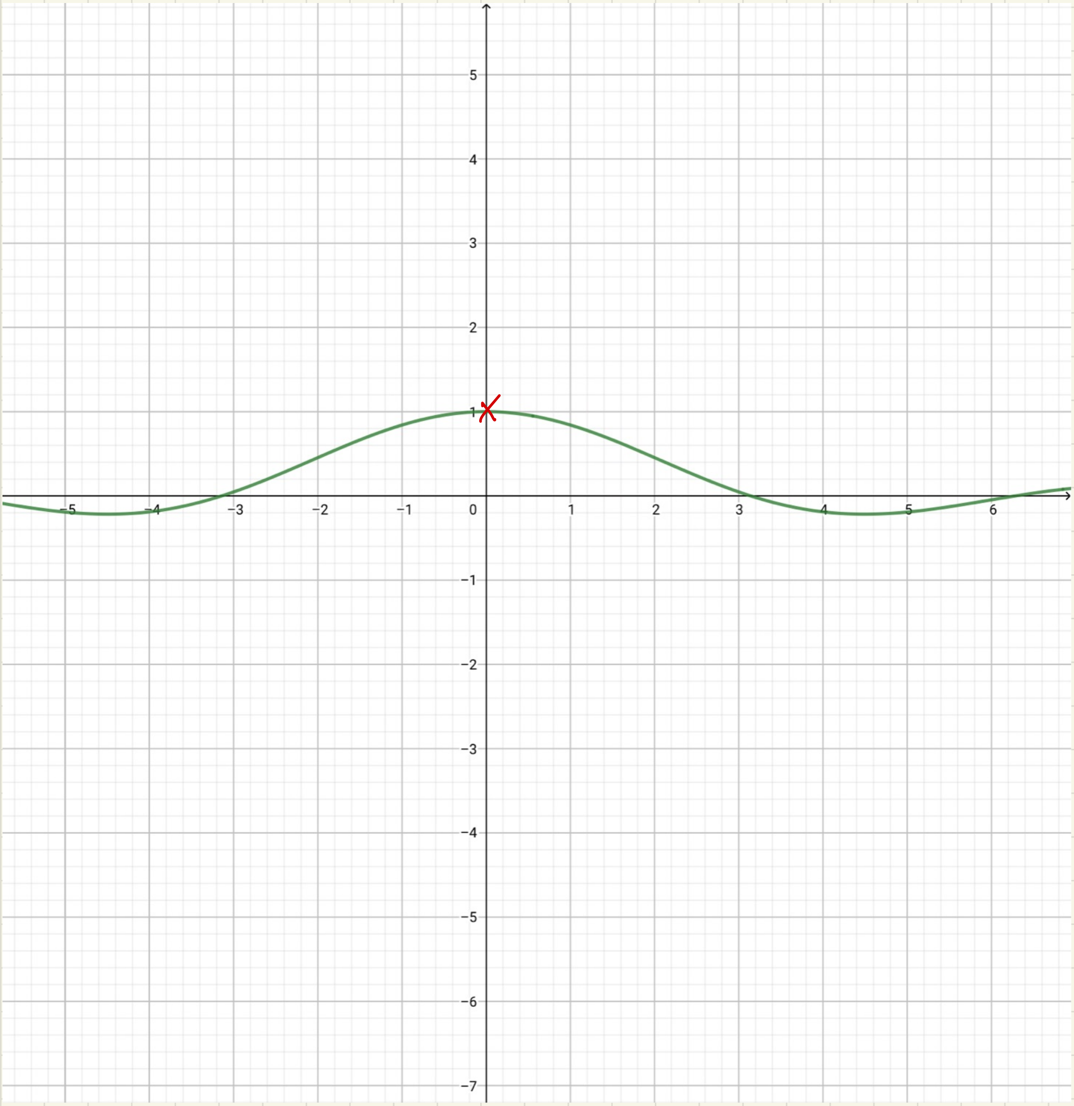
Provare che:

$$\lim_{x \to 1} \frac{1}{x^2 - 2x + 1} = +\infty$$

$$\lim_{x \to 2} \frac{1}{4x - x^2 - 4} = -\infty$$

$$\lim_{x \to x_0} f(x) = \begin{cases} l \in \mathbb{R} \\ +\infty \\ -\infty \end{cases}$$





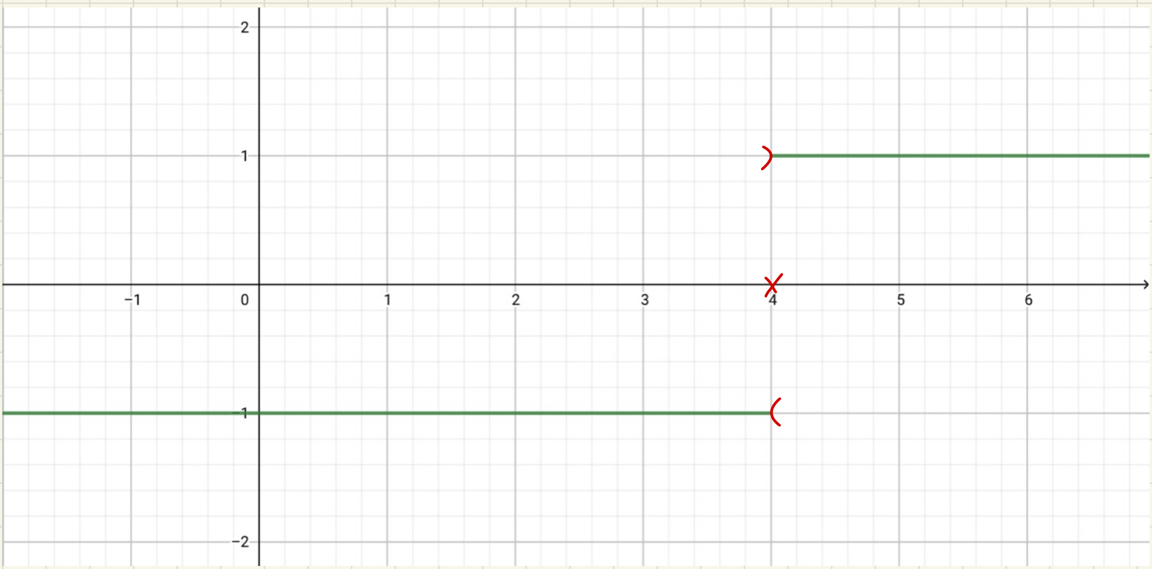
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

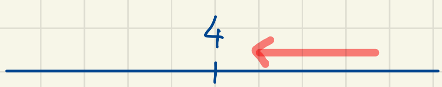
LIMITE DESTRO E SINISTRO:

Esempio:

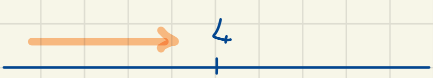
$$f(x) = \frac{x-4}{|x-4|} = \begin{cases} 1 & \text{se} \quad x > 4 \\ -1 & \text{se} \quad x < 4 \end{cases}$$

$$D(f) = \mathbb{R} \setminus \{4\}$$





$f(x) \longrightarrow 1$



$f(x) \longrightarrow -1$

Talvolta è necessario distinguere come ci si avvicina a 4.

DEF. (limite destro, sinistro; caso finito)

$f : A \longrightarrow \mathbb{R}$,

x_0 punto di accumulazione di A

$\ell \in \mathbb{R}$

$$\lim_{x \to x_0^+} f(x) = \ell \iff \forall \varepsilon > 0,\ \exists\, \delta = \delta(x_0, \varepsilon) > 0 :$$

(limite destro)

$$\forall x \in A : \quad x_0 < x < x_0 + \delta$$

$$\Longrightarrow \quad |f(x) - \ell| < \varepsilon$$

$$\lim_{x \to x_0^-} f(x) = \ell \iff \forall \varepsilon > 0,\ \exists\, \delta = \delta(x_0, \varepsilon) > 0 :$$

(limite sinistro)

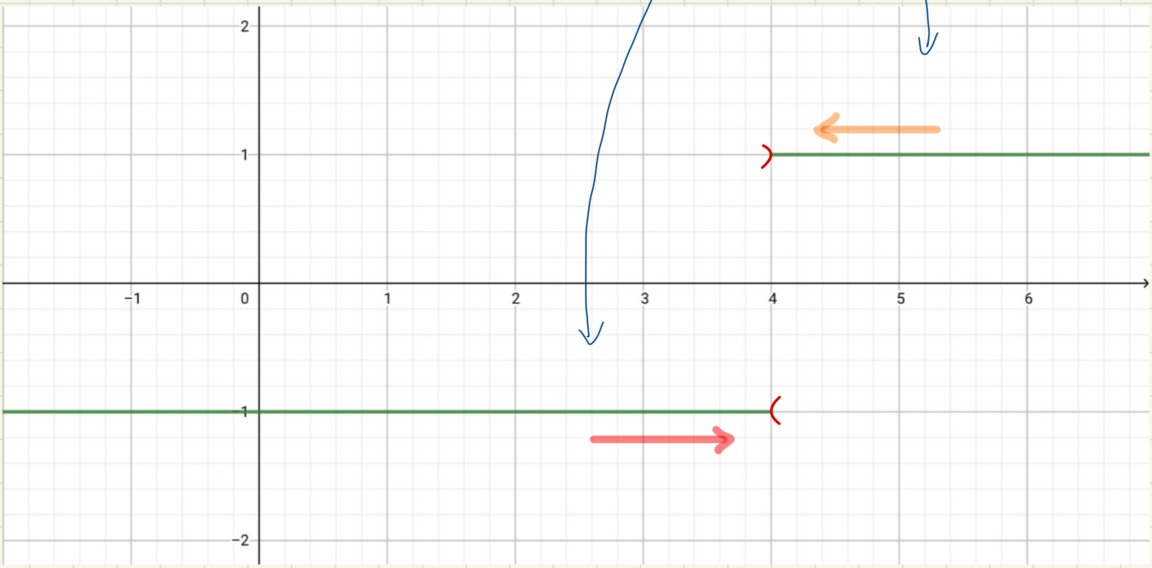
$$\forall x \in A : \quad x_0 - \delta < x < x_0$$

$$\Longrightarrow \quad |f(x) - \ell| < \varepsilon$$

Nell' esempio precedente:

$$\lim_{x \to 4^-} \frac{x-4}{|x-4|} = -1$$

$$\lim_{x \to 4^+} \frac{x-4}{|x-4|} = 1$$

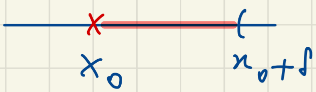


DEF. (limite destro, sinistro; caso infinito)

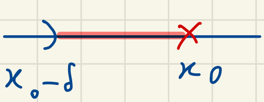
$f: A \longrightarrow \mathbb{R}$,

x_0 punto di accumulazione di A

$$\lim_{x \to x_0^+} f(x) = +\infty \ (-\infty) \iff \forall M, \ \exists \delta = \delta(x_0, M) > 0 : \forall x \in A : \ x_0 < x < x_0 + \delta \implies f(x) > M \ \ (f(x) < M)$$

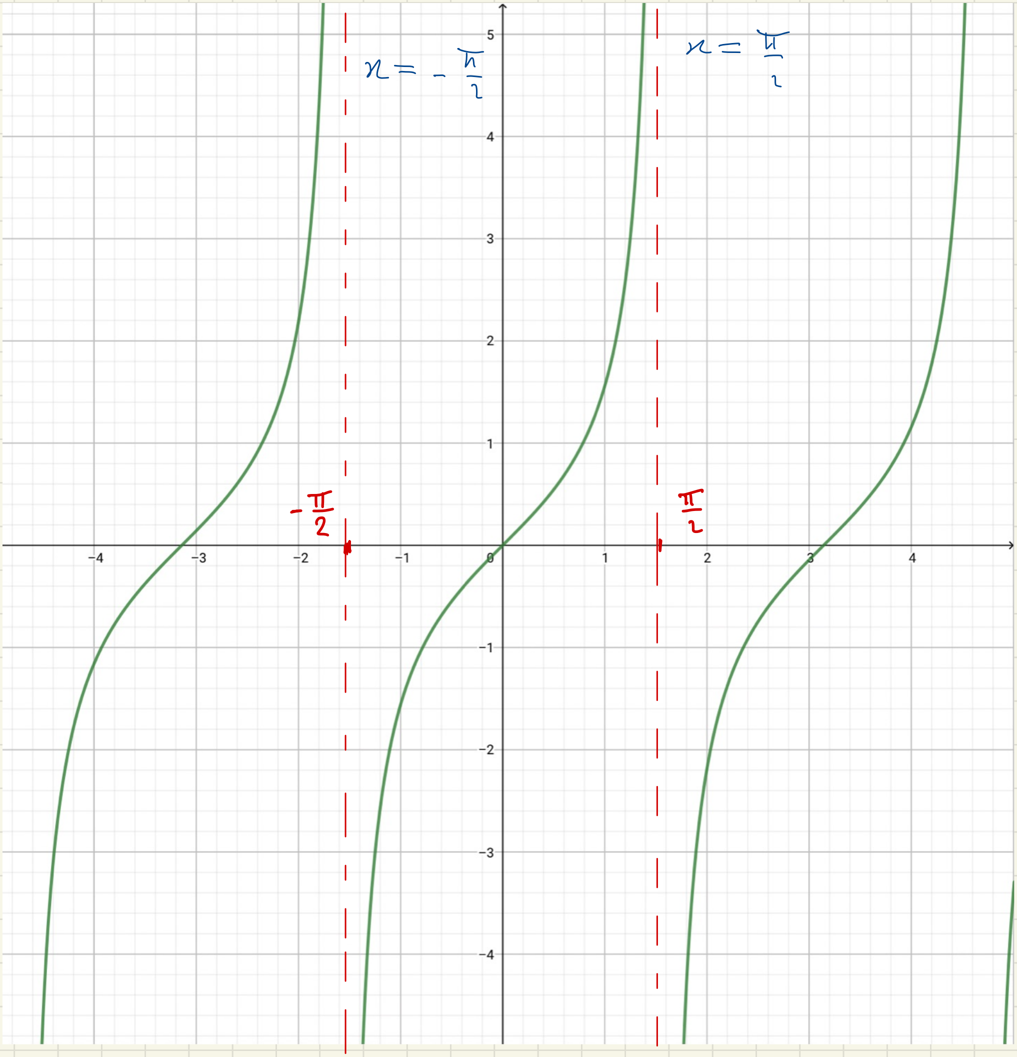


$$\lim_{x \to x_0^-} f(x) = +\infty \ (-\infty) \iff \forall M, \ \exists \delta = \delta(x_0, M) > 0 : \forall x \in A : \ x_0 - \delta < x < x_0 \implies f(x) > M \ \ (f(x) < M)$$



$x = x_0$ si dice ASINTOTO VERTICALE

Esempio: $y = \operatorname{tg} x$



$$\lim_{x \to \frac{\pi}{2}^-} \operatorname{tg} x = +\infty \qquad \lim_{x \to \frac{\pi}{2}^+} \operatorname{tg} x = -\infty$$

$$\lim_{x \to -\frac{\pi}{2}^-} \operatorname{tg} x = +\infty \qquad \lim_{x \to -\frac{\pi}{2}^+} \operatorname{tg} x = -\infty$$