Dir. di max crescita

$A \subseteq \mathbb{R}^2$ aperto, $f : A \longrightarrow \mathbb{R}$. f diff. in $(\bar{x},\bar{y}) \in A$

allora $$\max_{\substack{v \in \mathbb{R}^2 \\ |v|=1}} \frac{\partial f}{\partial v}(\bar{x},\bar{y}) = \frac{\partial f}{\partial v_{max}}(\bar{x},\bar{y}),$$

con $$v_{max} = \frac{\nabla f(\bar{x},\bar{y})}{|\nabla f(\bar{x},\bar{y})|}.$$ Inoltre

$$\frac{\partial f}{\partial v_{max}}(\bar{x},\bar{y}) = |\nabla f(\bar{x},\bar{y})|$$

Derivate funzioni composte

Caso modello: (derivate lungo una curva)

Premessa: curve (cammini) in $\mathbb{R}^n$

(·) $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ (scalari)

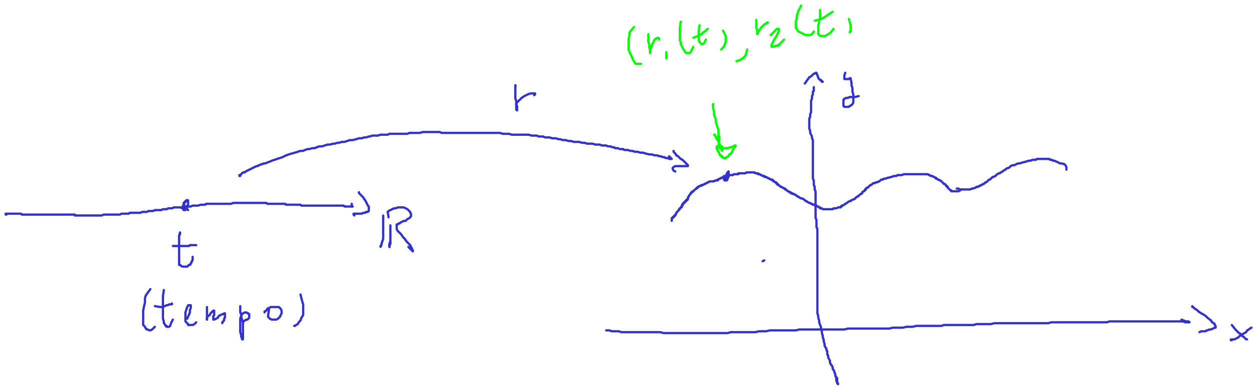
(··) $r :]a,b[\longrightarrow \mathbb{R}^n$ (cammini parametrizzati o curve) $\quad]a,b[\subseteq \mathbb{R}^1$

Considero $r :]a,b[\longrightarrow \mathbb{R}^n$

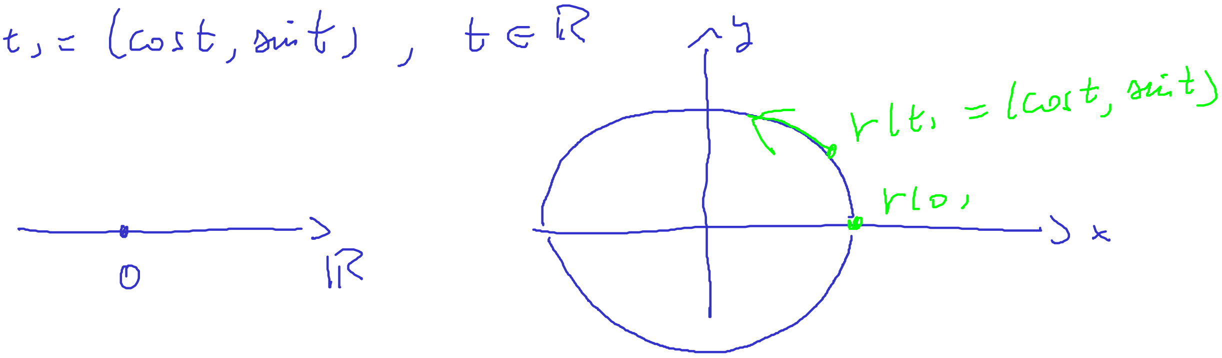
$]a,b[\ni t \longmapsto r(t) = (r_1(t), r_2(t), \dots, r_n(t)) \in \mathbb{R}^n$

ES(1) $r : \mathbb{R} \longrightarrow \mathbb{R}^2$, $r(t) = (\cos t, \sin t)$

$r_j :]a,b[\longrightarrow \mathbb{R}$ funzione scalare



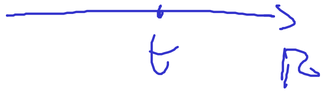
ES: $r(t) = (\cos t, \sin t)$, $t \in \mathbb{R}$

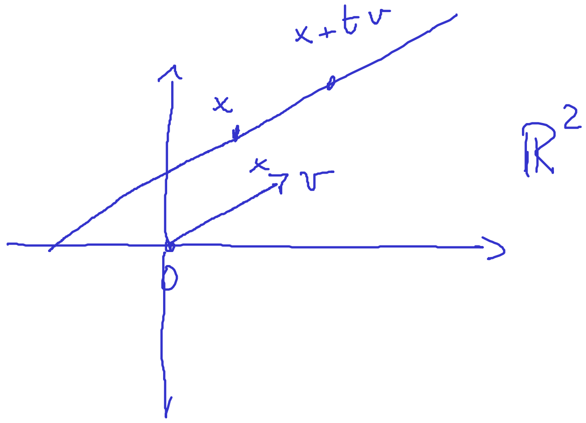


ES 2 $\mathbb{R}^n$ $\quad x \in \mathbb{R}^n$, $v \neq 0$ in $\mathbb{R}^n$

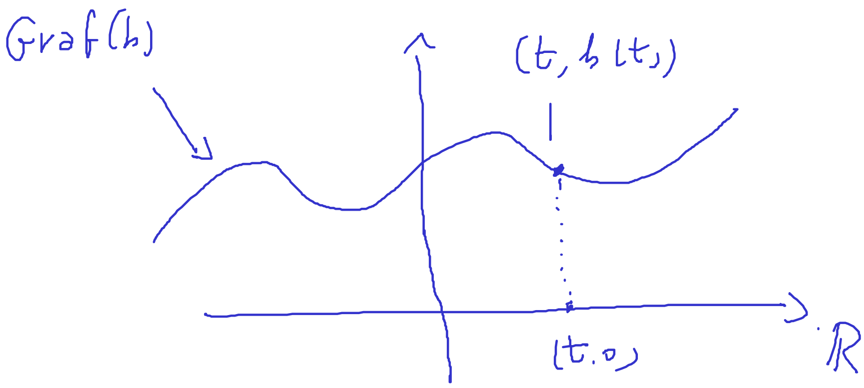
$r : \mathbb{R} \to \mathbb{R}^n$, $\quad r(t) = x + tv$

percorso "rettilineo")



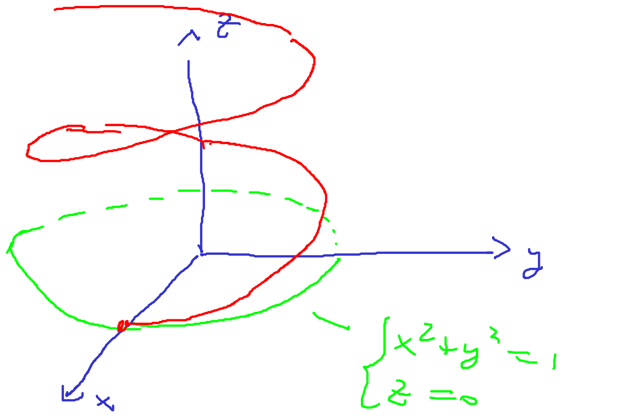


ES 3 $h : \mathbb{R} \to \mathbb{R}$, $\quad r : \mathbb{R} \to \mathbb{R}^2$, $\quad r(t) = (t, h(t)) \in \mathbb{R}^2$



ES 4 $r : \mathbb{R} \to \mathbb{R}^3$, $\quad r(t) = (\cos t, \sin t, t) \in \mathbb{R}^3$

$t > 0$ $\qquad$ elica



Def (velocità di un cammino) $r :]a,b[\to \mathbb{R}^n$ ($r(t) = (r_1(t), \dots, r_n(t))$). Sia $t \in]a,b[$. Se le funzioni $r_1, \dots, r_n$ sono derivabili in t, si dice che r è derivabile in t e si pone $r'(t) = (r_1'(t), r_2'(t), \dots, r_n'(t))$ = velocità di r al tempo t.

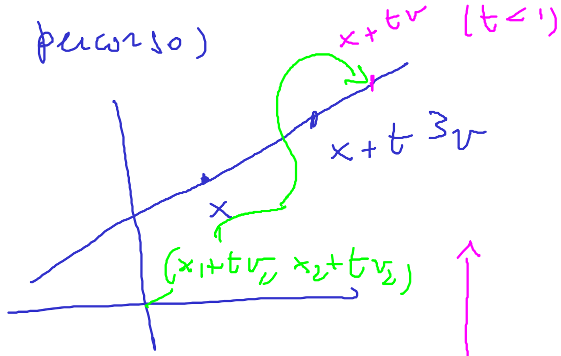
ES 2 $r(t) = x + tv = (x_1 + tv_1, x_2 + tv_2, \dots, x_n + tv_n)$

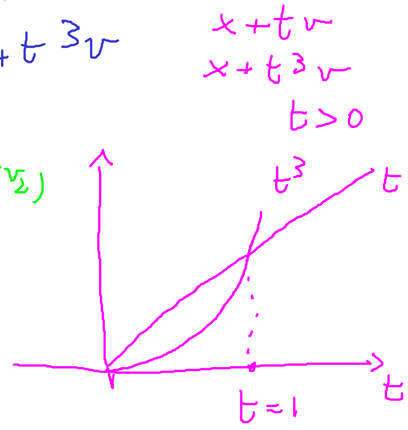
$r'(t) = (v_1, v_2, \dots, v_n) = v$ (costante in t)

ES 2 bis $\rho(t) = x + t^3 v$ (stesso percorso)

$\rho'(t) = 3t^2 v \in \text{span}\{v\} \ \forall t$

dipende da t





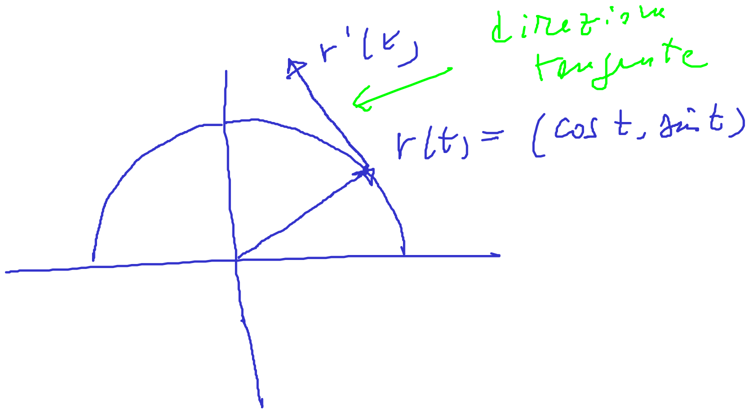
$\{r(t) / t \in (-\infty, +\infty)\} =$

$= \{\rho(t) / t \in \mathbb{R}\}$ (stesse immagine)

Def: (velocità scalare di $r:]a,b[\longrightarrow \mathbb{R}^n$)
Se r è derivabile in $t \in]a,b[$, $r'(t)$ velocità,
$\|r'(t)\|$ = velocità scalare

ES 1 $r(t) = (\cos t, \sin t)$, $r'(t) = (-\sin t, \cos t)$

$r(t) \perp r'(t)$



Formula Taylor

Sia $r:]a,b[\longrightarrow \mathbb{R}^n$, derivabile in $t \in]a,b[$. Vale dunque

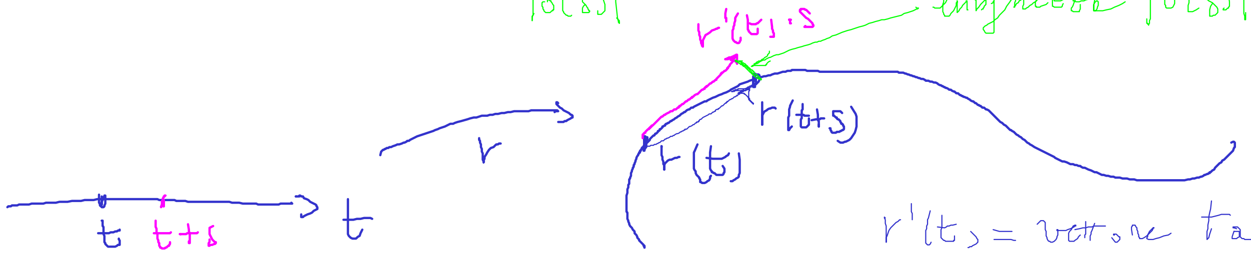
$$\begin{cases} r_1(t+s) = r_1(t) + r_1'(t)s + o_1(s), \quad s \to 0 \\ \vdots \\ r_n(t+s) - r_n(t) = r_n'(t)s + o_n(s) \end{cases}$$

$r'(t) \neq 0$

$o(s) = (o_1(s), \dots, o_n(s))$

$r(t+s) - r(t) = r'(t)s + o(s)$ $\left(\lim_{s \to 0} \frac{|o(s)|}{s} = 0\right)$

$|o(s)|$



$r'(t)$ = vettore tangente

ES (curve singolari) $r(t) = (t^3, t^2)$ ⟵ | regolari

$r'(t) = (3t^2, 2t)$ $\forall t \in \mathbb{R}$

$r'(0) = (0,0)$ $t=0$ "punto singolare"

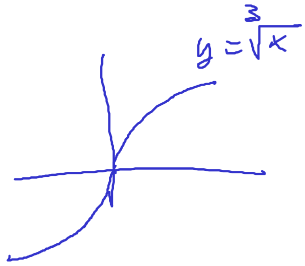
$r(t) = (t^3, t^2)$ regolare $t^3 = s$ (nuova variabile)

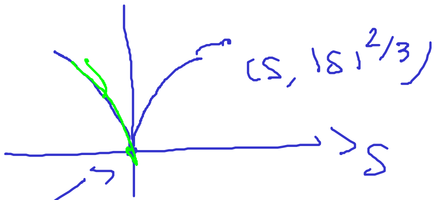
$t = s^{1/3} =$ (con segno)

$t^2 = (s^{1/3})^2 = |s|^{2/3}$

↳ ($t^2 \geq 0 \ \forall t$)

$\rho(s) = (s, |s|^{2/3})$ → curva con stesso percorso di r





$\rho(0) = r(0)$ (punto singolare)

| $\rho'(0)$ non esiste

| **Def** $r:]a,b[\to \mathbb{R}^n$, $r'(t) =$ velocità

| se $\exists\, r_j''(t)$ $\forall j = 1, \dots, n$, poniamo

| $r''(t) = (r_1''(t), \dots, r_n''(t))$ vettore accelerazione

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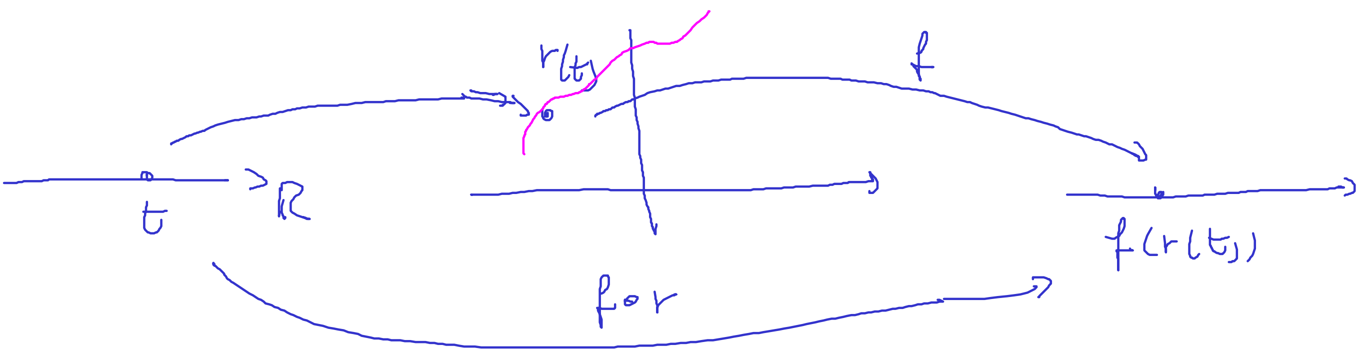
Derivata lungo una curva (cammino)

Sia $r:]a,b[\to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$
(A) (A)



$(f \circ r)(t) = f(r(t))$, r derivabile in t
f diff. in r(t)

ES $r(t) = (2t, \cos t)$, $f(x,y) = x^2 e^{2y}$

$$(f\circ r)(t) = f(r(t)) = (2t)^2 e^{2\cos t} = 4t^2 e^{2\cos t}$$

$$\frac{d}{dt}(f\circ r)(t) = 8t\, e^{2\cos t} - 8t^2 \sin t\, e^{2\cos t}$$

Teorema $r:]a,b[\to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ diff in $r(t)$

derivabile in $t \in]a,b[$.

Allora $(f\circ r)'(t) = \langle \nabla f(r(t)), r'(t) \rangle$

$$= \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(r(t))\, r_k'(t)$$

ES $r(t) = (2t, \cos t)$, $f(x,y) = x^2 e^{2y}$

$\nabla f(x,y) = (2x e^{2y}, 2x^2 e^{2y})$, $r'(t) = (2, -\sin t)$

$$(f\circ r)'(t) = \langle \nabla f(\underbrace{2t, \cos t}_{r(t)}), (\underbrace{2, -\sin t}_{r'(t)}) \rangle$$

$$= \langle (2\cdot(2t)\, e^{2\cos t}, 2(2t)^2 e^{2\cos t}), (2, -\sin t) \rangle$$

$$= 8t\, e^{2\cos t} - 8\sin(t)\, t^2 e^{2\cos t}$$

Dim $r:]a,b[\to \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$, differenziabili

$$\underbrace{\lim_{s\to 0} \frac{(f\circ r)(t+s) - (f\circ r)(t)}{s}}_{(def)\ (f\circ r)'(t)} = \langle \nabla f(r(t)), r'(t) \rangle$$

$$f(\underbrace{r(t+s)}_{x}) - f(\underbrace{r(t)}_{\bar{x}}) \underset{\text{Taylor}}{=} \langle \nabla f(r(t)), \underbrace{r(t+s) - r(t)}_{x-\bar{x}} \rangle + o(\underbrace{\|r(t+s) - r(t)\|}_{|x-\bar{x}|})$$

$r(t+s) - r(t) = r'(t)s + o(s)$

$$= (1) + (2)$$

$$\frac{(1)}{s} = \frac{1}{s} \langle \nabla f(r(t)), r'(t)s + o(s) \rangle = \langle \nabla f(r(t)), \frac{r'(t)s}{s} \rangle + \langle \nabla f(r(t)), \frac{o(s)}{s} \rangle$$

$$= \underbrace{\langle \nabla f(r(t)), r'(t) \rangle}_{\text{costante in } s} + \langle \underbrace{\nabla f(r(t))}_{\text{cost. in } s}, \underbrace{\frac{o(s)}{s}}_{\to 0} \rangle \xrightarrow[s\to 0]{}$$

$\xrightarrow[s \to 0]{} \langle \nabla f(r(t)), r'(t) \rangle$

(2) (informale) $o(|r(t+s) - r(t)|) = o(|\underbrace{r'(t)s + o(s)}_{\approx s}|) =$

$= o(s)$ ((verifiche formali da fare))

$\longrightarrow \dfrac{o(|r(t+s) - r(t)|)}{s} \xrightarrow[s \to 0]{} 0$

ES $f(x,y) = \ln(1 + x^2 + xy)$ $\quad r(t) = (2t, e^{-t})$

Calcolo $(f \circ r)'(t)$ $\forall t$

$\nabla f(x,y) = \left(\dfrac{2x+y}{1+x^2+xy}, \dfrac{x}{1+x^2+xy} \right)$, $\quad r'(t) = (2, -e^{-t})$

$(f \circ r)'(t) = \left\langle \left(\dfrac{4t + e^{-t}}{1+4t^2+2te^{-t}}, \dfrac{2t}{1+4t^2+2te^{-t}} \right), (2, -e^{-t}) \right\rangle$

$= \dfrac{1}{1+4t^2+2te^{-t}} \left\{ (4t + e^{-t})2 + 2t(-e^{-t}) \right\} = \dots$

A casa: scrivere $(f \circ r)(t)$ e derivare $\leftarrow$

ES Sapendo che $\nabla f(x,y) = \left(\dfrac{y}{(x+y)^2}, \dfrac{-x}{(x+y)^2} \right)$

Calcolare $\dfrac{d}{dt} f(t^2, e^{-t}) = (\cdot)$ $\quad$ (f non nota)

$r(t) = (t^2, e^{-t})$

$(\cdot) = \dfrac{d}{dt}(f \circ r)(t) = \langle \nabla f(r(t)), r'(t) \rangle$

$r'(t) = (2t, -e^{-t})$

$(f \circ r)'(t) = \left\langle \left(\dfrac{e^{-t}}{(t^2+e^{-t})^2}, \dfrac{-t^2}{(t^2+e^{-t})^2} \right), (2t, -e^{-t}) \right\rangle$

$= \dfrac{(2t + t^2)e^{-t}}{(t^2+e^{-t})^2}$

Gradiente – insiemi di livello (di f scalare)

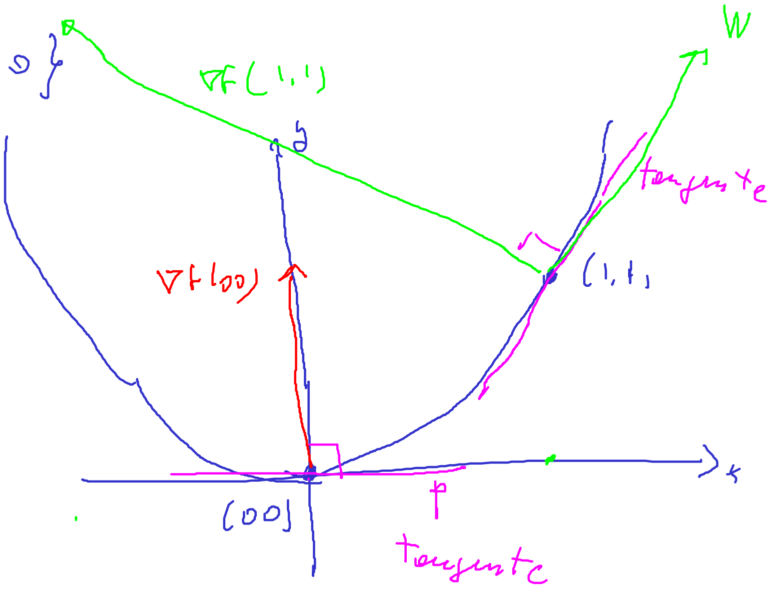
ES $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = y - x^2$

$L_0 := \{(x,y) \in \mathbb{R}^2 \,/\, f(x,y) = 0\}$

$\Updownarrow$

$y = x^2$

$L_0 \longrightarrow$



$(0,0) \in L_0$

$(1,1) \in L_0$

$\nabla f(x,y) = (-2x, 1)$

$\nabla f(0,0) = (0,1)$

$\nabla f(1,1) = (-2,1)$

Verifica che $(-2,1) \perp$ retta tangent

$y = x^2 \quad y' = 2x\Big|_{x=1} = 2$

Eq retta tg al grafico della parabola in $(1,1)$

$y = 2(x-1) + 1 = 2x - 1$

$w = (1,2)$ (direzione della retta)

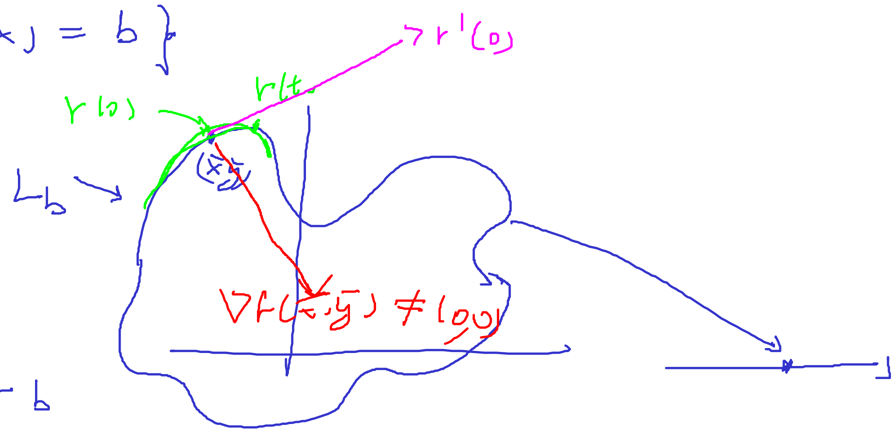
$\langle w, \nabla f(1,1) \rangle = \langle (1,2), (-2,1) \rangle = 0$

Questo fenomeno di ortogonalità è generale

Se $f : \mathbb{R}^2 \to \mathbb{R}$ è diff., considero

$L_b = \{x \in \mathbb{R}^2 \,/\, f(x) = b\}$

$(\bar{x}, \bar{y}) \in L_b$



Se f è regolare si può costruire una curva non singolare $r :]-1,1[\longrightarrow L_b$ che soddisfi $r(0) = (\bar{x}, \bar{y})$

Tale curva soddisfa $f(r(t)) = b \quad \forall\, t \in]-1,1[$

$\Rightarrow (f \circ r)'(t) = 0 \quad \forall t$

$\Rightarrow \langle \nabla f(r(t)), r'(t) \rangle = 0 \quad \forall t \in]-1,1[$

se $r(0) = (\bar{x}, \bar{y})$, con $t = 0$ trovo

$\langle \nabla f(\bar{x}, \bar{y}), \underline{r'(0)} \rangle = 0$

direzione tangente ad L_b